Karlsruhe Institute of Technology (KIT)
Institut für Analysis
Prof. Dr. Michael Plum
Dipl.-Math.techn. Rainer Mandel

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Nonlinear Boundary Value Problems: 5th problem sheet

Exercise 13

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let H be one of the spaces $H^1(\Omega), H_0^1(\Omega)$. Prove:

a) If $u \in H$ then $|u|, u^+, u^- \in H$ where $u^+ = \max\{u, 0\}, u^- = -\min\{-u, 0\}$. Determine the corresponding weak derivatives.

b) If $u \in H^1(\Omega)$ and $u \leq 0$ on $\partial\Omega$ in the trace sense then $u^+ \in H_0^1(\Omega)$.

Generalise the result obtained in a) to unbounded domains $\Omega \subset \mathbb{R}^n$.

Hint: In a) the functions $f_\varepsilon(z) = \sqrt{z^2 + \varepsilon^2} - \varepsilon$ might be useful. In b) you may use that $v \in H_0^1(\Omega)$ if and only if $\gamma(v) = 0$ where $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ denotes the trace map.

Exercise 14

Consider on $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ the boundary value problem

$$-\Delta u = 1 + u^2 \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega.$$

Prove existence of a weak solution $u \in H_0^1(\Omega)$ by constructing suitable sub- and supersolutions of the given problem.

Exercise 15

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f \in C^1([0, \infty))$ be bounded. Moreover assume $f(0) = 0$ and $f'(0) > \lambda_1$. Here, λ_1 denotes the principal (i.e. smallest) eigenvalue of $-\Delta$ on $H_0^1(\Omega)$, i.e. there is a $w \in H_0^1(\Omega)$ such that $-\Delta w = \lambda_1 w$ in Ω and $w = 0$ on $\partial\Omega$. It is moreover known that $0 < w \leq \|w\|_\infty < \infty$. Use the method of sub-and supersolutions to prove existence of a positive solution of

$$-\Delta u = f(u) \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega.$$