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Nonlinear Boundary Value Problems: 6th problem sheet

Exercise 16

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f : [0,\infty) \to \mathbb{R}$ satisfy $|f(z)| \le c|z|^{\beta+1}$ where $\beta > 0$. Assume that $u \in C^2(\overline{\Omega} \times [0,T])$ is a positive solution of

$$u_t - \Delta u = f(u) \text{ in } \Omega, \quad u(\cdot,t)|_{\partial\Omega} = 0$$

which has a blow-up at time $T > 0$ given by $\lim_{t\to T^-} \int_0^t \|u(\cdot,s)\|_\infty^\beta \, ds = \infty$. Prove the following statements:

a) $\psi_p'(t) \le 2cp\|u(\cdot,t)\|_\infty^\beta \psi_p(t)$ for $\psi_p(t) := \int_\Omega u(x,t)^{2p}\, dx$.

b) $\psi_p(t) \le \psi_p(0) \exp\left(2cp \int_0^t \|u(\cdot,s)\|_\infty^\beta \, ds\right)$.

c) $\|u(\cdot,t)\|_\infty \le \|u(\cdot,0)\|_\infty \exp\left(c \int_0^t \|u(\cdot,s)\|_\infty^\beta \, ds\right)$.

Derive $T \ge (c\beta\|u(\cdot,0)\|_\infty^\beta)^{-1}$.

Exercise 17

Let $f \in C^1(\mathbb{R}^n \times \mathbb{R})$ and $u \in C^2(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$ a classical solution of

$$-\Delta u = f(x,u) \quad \text{in } \mathbb{R}^n.$$

a) Give sufficient conditions on f which ensure $F(\cdot,u), G(\cdot,u) \in L^1(\mathbb{R}^n)$ for all $u \in C^2(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$ where

$$F(x,z) = \int_0^z f(x,s)\, ds, \qquad G(x,z) = \sum_{i=1}^n x_i \frac{\partial F}{\partial x_i}(x,z).$$

b) Assuming now that $F(\cdot, u), G(\cdot, u) \in L^1(\mathbb{R}^n)$ prove the following version of the Pohozaev identity:

$$\frac{n-2}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx = \int_{R^n} \big(nF(x, u(x)) + G(x, u(x))\big) \, dx$$

c) Consider the differential equation

$$-\Delta u = \lambda |u|^{p-2} u + \mu |u|^{q-2} u \quad \text{in } \mathbb{R}^n \tag{$*$}$$

where $p, q > 1$. Test the equation with u and use b) to derive conditions on p, q, λ, μ ensuring nonexistence of nontrivial classical solutions $u \in C^2(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$ of $(*)$.