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Nonlinear Boundary Value Problems: 7th problem sheet

Exercise 17

Consider the nonlinear boundary value problem

$$-\Delta u = |u|^{p-1}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{$*$}$$

for a bounded domain $\Omega \subset \mathbb{R}^n, n \geq 3$ not necessarily star-shaped. We will derive an upper bound for the critical exponent $p_c(\Omega)$ having the property that

$$p > p_c(\Omega) \quad \Longrightarrow \quad (*) \text{ has no nontrivial solutions in } C^2(\Omega) \cap C^1(\overline{\Omega}).$$

In the following let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of $(*)$ and $h \in C^1(\overline{\Omega}, \mathbb{R}^n)$.

a) Evaluate $\operatorname{div}\big(\frac{|\nabla u|^2}{2}h - \frac{|u|^{p+1}}{p+1}h - \langle h, \nabla u\rangle \nabla u\big)$ to prove the following generalisation of Pohozaev's identity:

$$\int_\Omega \Big(\frac{\operatorname{div}(h)}{2}|\nabla u|^2 - \frac{\operatorname{div}(h)}{p+1}|u|^{p+1} - \langle Dh(x)\nabla u, \nabla u\rangle\Big)\,dx = -\int_{\partial\Omega}\Big(\frac{|\nabla u|^2}{2} + \frac{|u|^{p+1}}{p+1}\Big)\langle h, \nu\rangle\,dS.$$

b) Now let h be a vectorfield in $\mathcal{H}(\Omega)$ where

$$\mathcal{H}(\Omega) = \{h \in C^1(\overline{\Omega}, \mathbb{R}^n) \;:\; \operatorname{div}(h) \equiv 1 \text{ in } \Omega,\ \langle h, \nu\rangle \geq 0 \text{ on } \partial\Omega\}$$

and set $\mu(h,x) := \sup\{\langle Dh(x)\xi, \xi\rangle : \xi \in \mathbb{R}^n, |\xi| = 1\}$. Show:

i) $\mu(h,x)$ is the largest eigenvalue of the matrix $A(x) := \frac{1}{2}(Dh(x) + Dh(x)^T)$.

ii) $\operatorname{tr}(A(x)) = 1$ and $\mu(h,x) \geq \frac{1}{n}$.

iii) If Ω is star-shaped then there exists $h \in \mathcal{H}(\Omega)$ such that $\mu(h, \cdot) \equiv \frac{1}{n}$.

c) Derive from a) the following inequality for arbitrary $h \in \mathcal{H}(\Omega), c \in \mathbb{R}$:

$$\frac{1 - 2\sup_{x\in\Omega}\mu(h,x) - c}{2}\int_\Omega |\nabla u|^2\,dx \leq \Big(\frac{1}{p+1} - \frac{c}{2}\Big)\int_\Omega |u|^{p+1}\,dx.$$

d) Set $M(\Omega) := \inf_{h\in\mathcal{H}(\Omega)} \sup_{x\in\Omega} \mu(h,x)$. Prove: If $M(\Omega) < \frac{1}{2}$ then the critical exponent $p_c(\Omega)$ exists and satisfies $p_c(\Omega) \le \frac{1+2M(\Omega)}{1-2M(\Omega)}$.

Finally consider the example of a *solid of revolution*: Let

$$\Omega_\phi = \{x \in \mathbb{R}^n : -1 < x_1 < 1, |x_2|^2 + \ldots + |x_n|^2 < \phi(x_1)\}, \quad S(\phi) := \sup_{t\in(-1,1)} \frac{t\phi'(t)}{\phi(t)}$$

where ϕ is assumed to be a smooth function so that Ω_ϕ is a C^1-domain. Show that $h_\delta(x) := (\delta x_1, \frac{1-\delta}{n-1}x_2, \ldots, \frac{1-\delta}{n-1}x_n) \in \mathcal{H}(\Omega)$ provided $\delta \in [0, \frac{2}{2+(n-1)S(\phi)}]$. Determine an upper bound for $p_c(\Omega)$.

Exercise 18

Let $\Omega \subset \mathbb{R}^n$ be an open domain which satisfies an interior ball condition and let

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u$$

be uniformly elliptic with continuous coefficients and $c \ge 0$. Prove:

a) *(Hopf's Lemma)* Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy

$$Lu \ge 0, \quad u(x_0) \le 0, \quad u(x_0) < u(x) \text{ for all } x \in \Omega,$$

for some $x_0 \in \partial\Omega$. Then $\frac{\partial u}{\partial \nu}(x_0) < 0$.

b) *(Strong minimum principle)* Assume that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies $Lu \ge 0$ in Ω and there is a $x_0 \in \Omega$ such that $u(x_0) = \min_{x\in\overline{\Omega}} u(x) \le 0$. Then u is constant.

c) Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy $Lu \ge 0$ in Ω and $u \ge 0$ on $\partial\Omega$. Deduce from a),b) the following alternative:

   i) $u = 0$ or

   ii) $u > 0$ in Ω and if $x_0 \in \partial\Omega, u(x_0) = 0$ then $\frac{\partial u}{\partial \nu}(x_0) < 0$.

Hints: In a) given the interior ball $B_R(z)$ for x_0 consider $v(x) = \delta(e^{-\alpha|x-z|^2} - e^{-\alpha|x-x_0|^2})$ on the annular region $A := B_R(z) \setminus \overline{B_r(z)}$. Choose $\alpha, r, \delta > 0$ in such a way that the weak maximum principle implies $u(x) - u(x_0) \le v(x)$ on $\overline{A}$ and conclude.
In b) consider the sets $\{x \in \Omega : u(x) = \inf_\Omega u\}, \{x \in \Omega : u(x) > \inf_\Omega u\}$.