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Nonlinear Boundary Value Problems: 8th problem sheet

Exercise 19

Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded domain and let $A \in C^1(\overline{\Omega}, \mathbb{R}^{n\times n}), b \in C(\overline{\Omega}, \mathbb{R}^n), c \in C(\overline{\Omega}, \mathbb{R})$ with $c \geq 0$ and $f \in C(\overline{\Omega} \times \mathbb{R})$. Moreover assume that there is a constant $a > 0$ such that $\xi^T A(x)\xi \geq a|\xi|^2$ for all $\xi \in \mathbb{R}^n$ and all $x \in \Omega$ and that there exists a function $\phi \in C^1(\overline{\Omega})$ such that $\nabla\phi(x) = A^{-1}(x)b(x)$.
Consider the nonlinear boundary value problem

$$\begin{aligned} -\operatorname{div}(A(x)\nabla u) + b(x)\cdot\nabla u + c(x)u &= f(x,u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Introduce a new unknown v by setting $u = e^{\psi}v$ to write (1) as the Euler-Lagrange equation of a variational problem.

Exercise 20

Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded domain, $\lambda > 0$ and $I : H_0^1(\Omega) \to \mathbb{R}$ be given by

$$I(u) = \int_\Omega \big(\frac{|\nabla u|^2}{2} + \lambda\cos(u)\big)\,dx \qquad (u \in H_0^1(\Omega)).$$

Prove:

a) I is weakly lower semicontinuous and I has a minimizer $u_0 \in H_0^1(\Omega)$.

b) There exists $\lambda_* > 0$ such that $u_0 = 0$ for all $0 < \lambda < \lambda_*$.

c) There exists $\lambda^* > 0$ such that $u_0 \neq 0$ for all $\lambda > \lambda^*$.

d) I is Fréchet-differentiable. Find the Euler-Lagrange equations for u_0.

Hints: In a) and c) Poincaré's inequality might be useful.