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Nonlinear Boundary Value Problems: 9th problem sheet

Exercise 21

Let $\alpha \in \mathbb{R}$ and consider the functionals

$$I : M_\alpha \to \mathbb{R}, u \mapsto \int_0^1 u(x)^2\,dx, \qquad \tilde{I} : \tilde{M}_\alpha \to \mathbb{R}, u \mapsto \int_0^1 u(x)^2\,dx$$

where the sets $M_\alpha, \tilde{M}_\alpha$ are given by $M_\alpha := \{u \in C[0,1] : u(0) = 0, u(1) = \alpha\}$ and $\tilde{M}_\alpha := \{u \in C_p[0,1] : u(0) = 0, u(1) = \alpha\}$. Here $C_p[0,1]$ denotes the space of piecewise continuous functions on $[0,1]$.

a) Prove that I is a convex and $\tilde{I}$ a strictly convex functional on its domain of definition. Conclude that there exists at most one minimizer of I.

b) For which values of α do $I, \tilde{I}$ have minimizers?

Exercise 22

Let $\Omega \subset \mathbb{R}^n$ be an non-empty bounded open set and let $f \in L^2(\Omega), V \in L^\infty(\Omega), V \geq 0$ and $\alpha \in \mathbb{R}$. Prove that the functional $I : H_0^1(\Omega) \to \mathbb{R}$ given by

$$I(u) = \int_\Omega \Big(\frac{1}{2}(|\nabla u|^2 + V(x)|u|^2) + \alpha\sqrt{|u|^2+1} - f(x)u\Big)\,dx \quad (u \in H_0^1(\Omega))$$

has a minimizer u_0 on $H_0^1(\Omega)$. Determine the Euler-Lagrange-equation for u_0.

Exercise 23

Let $\Omega \subset \mathbb{R}^n$ be non-empty and bounded. Moreover let $p \in [1, 2^*)$ where $2^* = \frac{2n}{n-2}$ for $n \geq 3$ and $2^* = \infty$ if $n = 1, 2$. Consider the functional $I : H_0^1(\Omega) \to \mathbb{R}$ given by

$$I(u) = \int_\Omega |u|^p \, dx \qquad (u \in H_0^1(\Omega))$$

on the set

$$V = \{u \in H_0^1(\Omega) : \int_\Omega |\nabla u|^2 \, dx = 1\}.$$

Prove:

i) $\inf_V I = 0$ and no minimizer exists.

ii) $\sup_V I < \infty$ and a maximizer exists.