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Nonlinear Boundary Value Problems: 10th problem sheet

Exercise 24

Consider the following elliptic system on the set $\Omega := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$:

$$
\begin{aligned}
-\Delta u &= \frac{2v + 2u}{1 + |u|^2 + |v|^2}(|\nabla u|^2 + |\nabla v|^2) \\
-\Delta v &= \frac{2v - 2u}{1 + |u|^2 + |v|^2}(|\nabla u|^2 + |\nabla v|^2).
\end{aligned} \tag{1}
$$

Show that the functions

$$
u(x) = \sin(\log(\log(|x|^{-1}))), \qquad v(x) = \cos(\log(\log(|x|^{-1})))
$$

define a discontinuous bounded weak solution of (1)[1].

Exercise 25

Let $B_1(0)$ denote the unit ball in $\mathbb{R}^n$ for $n \geq 3$. Prove that the function $u(x) := \frac{x}{|x|}$ is a discontinuous bounded weak solution of the elliptic system

$$
-\Delta u_i = u_i \sum_{j=1}^{n} |\nabla u_j|^2 \qquad \text{in } B_1(0), i = 1, \dots, n.
$$

Why is this result "restricted" to the case $n \geq 3$?

[1] This example is due to J.Frehse (1973).

Exercise 26

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2-domain, $f \in L^2(\Omega)$ and let $a_{ij} \in C^1(\overline{\Omega})$, $b_i, c \in L^\infty(\Omega)$ with $\xi^T A(x)\xi \geq \lambda|\xi|^2$ for the matrix $A = (a_{ij})$, $\lambda > 0$. Our aim is to show that every weak solution $u \in H_0^1(\Omega)$ of the elliptic problem

$$-\operatorname{div}(A\nabla u) + b \cdot \nabla u + cu = f \quad \text{in } \Omega$$

lies in fact in $H^2(\Omega)$ and not only in $H^2_{loc}(\Omega)$. In the following C denotes a constant independent of $u, \tilde{u}$.

a) By definition of a C^2-domain there is an open neighbourhood U of $x_0 \in \partial\Omega$, an open neighbourhood V of 0 and C^2-diffeomorphisms $\Psi : U \to V$, $\Phi := \Psi^{-1} : V \to U$ such that $\Psi(U \cap \Omega) = V \cap \{y_n > 0\}$ and $|\det(D\Phi)| = |\det(D\Psi)| = 1$.
Show that the function $\tilde{u} := u \circ \Phi$ is a weak solution of

$$-\operatorname{div}(\tilde{A}\nabla v) = \tilde{g} := \tilde{f} - \tilde{b} \cdot \nabla v - \tilde{c}v \quad \text{in } V \cap \{y_n > 0\}$$

where the parameters $\tilde{A}, \tilde{b}, \tilde{c}, \tilde{f}$ are given by

$$\tilde{A}(y) = D\Psi(\Phi(y))A(\Phi(y))D\Psi(\Phi(y))^T, \qquad \tilde{b}(y) = D\Psi(\Phi(y))b(\Phi(y)),$$
$$\tilde{c}(y) = c(\Phi(y)), \qquad \tilde{f}(y) = f(\Phi(y)).$$

Show that $\tilde{A}, \tilde{b}, \tilde{c}, \tilde{f}$ satisfy the same structural conditions as A, b, c, f (smoothness, boundedness and ellipticity).

b) Since V is open there exists $r > 0$ such that $B_{2r}(0) \subset V$. Let $\varrho \in C_0^\infty(\mathbb{R}^n)$ such that $\varrho \equiv 1$ on $B_r(0)$ and $\varrho \equiv 0$ on $B_{\frac{3}{2}r}(0)^c$. Show $-\nabla_{i,-h}(\varrho^2\nabla_{i,h}\tilde{u}) \in H_0^1(V \cap \{y_n > 0\})$ for $i = 1, \ldots, n-1$ provided $0 < |h| < \frac{r}{2}$.

c) Use the difference quotient technique to prove that the weak derivative $\partial_{ij}\tilde{u}$ exists in $L^2(B_r(0) \cap \{y_n > 0\})$ whenever $(i,j) \neq (n,n)$ and that we have the estimate

$$\|\partial_{ij}\tilde{u}\|_{L^2(B_r(0)\cap\{y_n>0\})} \leq C(\|\tilde{f}\|_{L^2(V\cap\{y_n>0\})} + \|\tilde{u}\|_{H^1(V\cap\{y_n>0\})}).$$

d) Use the (weak) differential equation in a) to prove $\tilde{u} \in H^2(B_r(0) \cap \{y_n > 0\})$ and

$$\|\tilde{u}\|_{H^2(B_r(0)\cap\{y_n>0\})} \leq C(\|\tilde{f}\|_{L^2(V\cap\{y_n>0\})} + \|\tilde{u}\|_{H^1(V\cap\{y_n>0\})}).$$

e) Use the estimate of d) to prove $u \in H^2(\Phi(B_r(0) \cap \{y_n > 0\}))$ and

$$\|u\|_{H^2(\Phi(B_r(0)\cap\{y_n>0\}))} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}).$$

f) Use the interior estimates of the lecture and a covering of $\partial\Omega$ by suitable open sets to conclude $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}).$$