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Nonlinear Boundary Value Problems: 11th problem sheet

Exercise 27

Let $n > 2$, $\Omega \subset \mathbb{R}^n$ a bounded domain and let $u \in H_0^1(\Omega)$, $u \geq 0$ be a positive weak solution of

$$-\operatorname{div}(A(x)\nabla u) + c(x)u = 0 \quad \text{in } \Omega$$

where $c \in L^\infty(\Omega)$ and A is assumed to be uniformly positive definite on Ω. We will show $u \in L^\infty(\Omega)$ using *Moser's iteration technique.*

a) Use Sobolev's imbedding theorem $H_0^1(\Omega) \to L^{\frac{2n}{n-2}}(\Omega)$ to derive the inequality

$$\Big(\int_\Omega u^{\frac{rn}{n-2}}\,dx\Big)^{\frac{n-2}{nr}} \leq C_1^{\frac{1}{r}}\Big(\frac{r^2}{r-1}\Big)^{\frac{1}{r}}\Big(\int_\Omega u^r\,dx\Big)^{\frac{1}{r}}$$

for all $r \geq 2$ where C_1 does not depend on r and u. To this end test the equation with $uu_m^{r-2} \in H_0^1(\Omega)$ where $u_m = \min\{u, m\}$ and send m to $+\infty$.

b) Set $\delta = \frac{n}{n-2}$ and $\Phi(s) := |\Omega|^{-1/s}\big(\int_\Omega u^s\,dx\big)^{1/s}$. Use a) to prove the inequality $\Phi(\delta r) \leq (2C_1|\Omega|^{2/n}r)^{1/r}\Phi(r)$ for all $r \geq 2$.

c) Show $\liminf_{k\to\infty}\Phi(\delta^k 2) \leq C_2\Phi(2)$ for some constant $C_2 > 0$ independent of u. Conclude $u \in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty(\Omega)} \leq C_2|\Omega|^{-1/2}\|u\|_{L^2(\Omega)}.$$

Hint: In a) use $\nabla(uu_m^{r-2}) = (\nabla u)u_m^{r-2} + (r-2)1_{\{u<m\}}(\nabla u)u^{r-2}$.

Exercise 28

Let Ω be a bounded domain and $\alpha \in \mathbb{R}$. Consider the functional

$$I : H_0^1(\Omega) \to \mathbb{R}, u \mapsto \int_\Omega \big(|\nabla u|^2 + u^2 - \alpha\cos(u)\sqrt{1+|u|^2}\big)\,dx.$$

Show that I has a minimizer on $M := \{v \in H_0^1(\Omega) : \int_\Omega |v|^2 + |v|\,dx = 1\}$.