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Nonlinear Boundary Value Problems: 12th problem sheet

Exercise 29

Let $\Omega \subset \mathbb{R}^n$ be a bounded nonempty domain and let $1 < p < \frac{n+2}{(n-2)_+}$.

a) Consider the nonlinear boundary value problem

$$-\Delta u = |u|^{p-1}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Find a functional $I : H_0^1(\Omega) \to \mathbb{R}$ and a subset $V \subset H_0^1(\Omega)$ such that minimization of I over V gives rise to a nontrivial weak solution of the above problem.

b) Which functional and which subset $V \subset H_0^1(\Omega)$ should be considered seeking for a weak solution of

$$-\operatorname{div}(A(x)\nabla u) + V(x)u = \Gamma(x)|u|^{p-1}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

when $A(x)$ is uniformly positive definite in Ω and $V, \Gamma \in L^\infty(\Omega)$ satisfy $V, \Gamma \geq 0$ almost everywhere and $\Gamma \not\equiv 0$?

Exercise 30

Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded domain and let $K \subset H_0^1(\Omega)$ be closed and convex. For a uniformly positive definite matrix $A \in L^\infty(\Omega)^{n\times n}$ and $c \in L^\infty(\Omega)$ such that $c(x) \geq c_0 > 0$ consider the functional

$$I : H_0^1(\Omega) \to \mathbb{R}, u \mapsto \int_\Omega \big(\frac{1}{2}\nabla u^T A(x)\nabla u + \frac{1}{2}c(x)u^2 - f(x)u\big)\,dx.$$

a) Assume that I has a minimizer u_0 over K. Find a variational inequality für u_0.

b) Determine constants $A, B > 0$ such that $I(u) \geq A\|u\|^2_{H_0^1(\Omega)} - B$.

c) Determine a radius $R > 0$ in terms of A, B such that the additional assumption $\{v \in H_0^1(\Omega) : \|v\|_{H_0^1(\Omega)} \leq R\} \subset K$ implies $u_0 \in \mathring{K}$. Which variational equation does u_0 satisfy in this case?