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Nonlinear Boundary Value Problems: 13th problem sheet

Exercise 31

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f \in L^2(\Omega)$. Show that there exists a unique minimizer u_0 of the functional

$$I(u) = \int_\Omega \big(\frac{1}{2}|\nabla u|^2 - f(x)u\big)\, dx \qquad \text{over } V := \{v \in H_0^1(\Omega) : |\nabla v| \leq 1 \text{ a.e.}\}$$

Prove that u_0 satsifies the variational inequalitiy

$$\int_\Omega \nabla u \cdot \nabla(v-u)\, dx \geq \int_\Omega f(v-u)\, dx \quad \forall v \in V.$$

Hint: To show that the candidate for a minimizer lies in V it is convenient to argue by contradiction. You might need the fact (to be proved!) that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ implies $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\Omega)^n$ for a subsequence.

Exercise 32

Check whether the following mappings satisfy the Palais-Smale condition:

i) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x\sin(x)$

ii) $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto \sin(x) + xy^2$

iii) $I : L^2([-1, 1]) \to \mathbb{R}$, $u \mapsto \int_0^1 u(t)^2\, dt$

iv) $I : H_0^1([-1, 1]) \to \mathbb{R}$, $u \mapsto \int_{-1}^1 u'(t)^2\, dt + \int_0^1 u(t)^2\, dt$

v) $I : H_0^1([-1, 1]) \to \mathbb{R}$, $u \mapsto \int_0^1 u'(t)^2\, dt + \int_0^1 u(t)^2\, dt$.

vi) $I : H_0^1(\Omega) \to \mathbb{R}$, $u \mapsto \int_\Omega \big(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\big)\, dx$ where $1 < p < \frac{n+2}{n-2}$ and Ω is a bounded domain.