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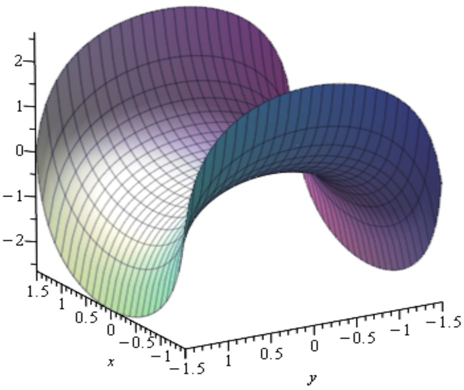
Nonlinear Boundary Value Problems

Exercise sheet 1

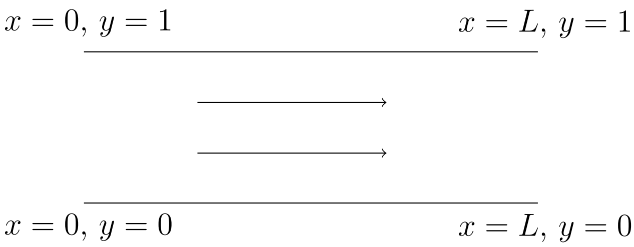
Exercise 1:
Show that the Scherk Surface, which can be represented as the graph of the function

$$\varphi : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R},\ \varphi(x, y) = \log(\cos(y)) - \log(\cos(x)),$$

is a minimal surface.



Exercise 2:
Consider the stationary flow of an incompressible fluid in a two dimensional pipe of length $L > 0$ and hight 1.



Find a solution of the Navier-Stokes-Equation (with viscosity term $\eta\Delta v$ and without any external force f) of the form

$$v(x, y) = (u(x, y), 0)$$

— please turn around —

satisfying the following boundary conditions

$$\begin{aligned} p(0,y) = p_1,\ p(L,y) = p_2 \qquad & \text{for all } y \in (0,1), \\ u(x,0) = u(x,1) = 0 \qquad & \text{for all } x \in (0,L), \end{aligned}$$

where v is the speed, p is the pressure and $p_1, p_2 \in \mathbb{R}$ are given.

Exercise 3:
Consider the Euler equation for a stationary, invicid ($\eta = 0$) flow of a gas. Let a pressure-density relation $p = \hat{p}(\rho)$ be given and assume that there are no external forces ($f = 0$).

A C^1-curve $\gamma : [a,b] \to \mathbb{R}$ is called *streamline* of the flow, if the relation $\gamma'(s) = v(\gamma(s))$ holds for all $s \in [a,b]$. Prove the existence of a function $w = w(\rho)$ satisfying:

$$\tfrac{1}{2}|v|^2 + w(\rho) \text{ is constant along any streamline.}$$

Hint: Use the identity

$$\frac{1}{2}\nabla|v|^2 = (v \cdot \nabla)v + v \times (\nabla \times v)$$

Exercise 4:
For given $\omega > 0$ find an explicit solution of the nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^2 u = 0$$

by making the ansatz:

$$u(x,t) = e^{i\omega t} v(x),$$

where v is a symmetric, positive function $v : \mathbb{R} \to \mathbb{R}$ with $\lim_{x \to \pm\infty} v(x) = 0 = \lim_{x \to \pm\infty} v'(x)$