Karlsruher Institut für Technologie (KIT)
Institut für Analysis
Prof. Dr. Michael Plum
M. Sc. Peter Rupp

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Nonlinear Boundary Value Problems

Exercise sheet 10

Exercise 31:
In this exercise we adapt the variational calculus for functionals acting on functions with values in multidimensional spaces. Let $\Omega \subset \mathbb{R}^n$ be a domain and $L : \mathbb{R}^{n\times m} \times \mathbb{R}^m \times \Omega \to \mathbb{R}$.

(a) Consider the functional

$$I(u) = \int_\Omega L(\nabla u, u, x)\, dx \qquad (u \in H_0^1(\Omega)^m)$$

and calculate the corrsponding system of Euler-Lagrange equations.

(b) Calculate the system of Euler-Lagrange equations to the following functional

$$I(u) = \int_\Omega \sum_{i,j=1}^{n} \left|\frac{\partial u_i}{\partial x_j}\right|^2 + \left[\sum_{i,j=1}^{n} \left((n-2)\delta_{ij} + n\frac{x_i x_j}{|x|^2}\right)\frac{\partial u_i}{\partial x_j}\right]^2 dx \qquad (u \in H^1(\Omega)^n).$$

Exercise 32:

(a) Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be open and bounded, $x_0 \in \Omega$, $u \in H^1(\Omega)^m \cap C^2(\Omega \setminus \{x_0\})^m$ and suppose that

$$-\sum_{i=1}^{m} div(A^{ji}(x)\nabla u_i(x)) = 0 \qquad (j = 1, ..m)$$

in the classical sense on $\Omega \setminus \{x_0\}$, where $A^{ji} \in C^1(\Omega \setminus \{x_0\})^{n\times n} \cap L^\infty(\Omega)^{n\times n}$. Prove that

$$\int_\Omega \sum_{i=1}^{m} (\nabla\phi(x))^T A^{ji}(x)\nabla u_i(x)\, dx = 0 \qquad (j = 1, ..m,\ \phi \in C_0^1(\Omega))$$

(b) Let $n \geq 3$. Show that u definied by

$$u(x) := x|x|^{-\gamma},$$

where $\gamma := \frac{n}{2}(1 - [(2n-2)^2 + 1]^{-1/2})$ is an element of $H^1(\{|x| < 1\})^n$ is an unbounded weak solution to the system of Euler-Lagrange equations calculated in Exercise 31.

— please turn over —

Exercise 33:

Let Ω be a domain in $\mathbb{R}^n$. In this exercise we discuss a technique of how to approximate functions belonging to some Sobolev spaces $H^{k,p}(\Omega)$ by smooth functions. Let $\rho \in C_c^\infty(B_1(0))$ be a standard mollifier, that is, $\rho(x) \in [0,1]$ for all $x \in B_1(0)$, $\rho(0) = 1$ and $\int_{B_1(0)} \rho(x)\,dx = 1$. Let $u \in L^p_{loc}(\Omega)$ for some $p \in [1,\infty)$. For a subdomain Ω_0 of Ω with $\overline{\Omega}_0 \subset \Omega$ and $0 < \varepsilon < dist(\Omega_0, \mathbb{R}^n \setminus \Omega)$ define $u_\varepsilon : \Omega_0 \to \mathbb{R}$ by

$$u_\varepsilon(x) := \frac{1}{\varepsilon^n} \int_\Omega \rho\left(\frac{x-y}{\varepsilon}\right) u(y)\,dy \qquad (x \in \Omega_0)$$

Prove the following statements:

(a) $u_\varepsilon \in C^\infty(\Omega_0)$

(b) $\|u_\varepsilon - u\|_{L^\infty(\Omega)} \to 0$ for $\varepsilon \to 0$, provided that $u \in C(\Omega)$.

(c) $\|u_\varepsilon - u\|_{L^p(\Omega)} \to 0$ for $\varepsilon \to 0$.

(d) $\|u_\varepsilon - u\|_{H^{k,p}(\Omega)} \to 0$ for $\varepsilon \to 0$, provided that $u \in H^{k,p}(\Omega)$.

We wish you a mery Christmas and a happy new year!