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Nonlinear Boundary Value Problems

Exercise sheet 11

In the following exercises let $\Omega \subset \mathbb{R}^n$ be a bounded domain.

Exercise 34:
Consider the functional $I : V \to \mathbb{R}$, $I(u) = \int_\Omega |u|^{p+1}\,dx$, where $p < \infty$ if $n = 1, 2$ and $p \leq \frac{n+2}{n-2}$ if $n \geq 3$ and $V \subset H_0^1(\Omega)$ is defined by

$$V := \{v \in H_0^1(\Omega) : \int_\Omega |\nabla v|^2 = 1\}.$$

Prove the following statements:

(a) $\inf\limits_V I = 0$, but there is no minimizer.

(b) $\sup\limits_V I < \infty$ and there is a maximizer.

Exercise 35:
For $\alpha \in \mathbb{R}$ consider the functional

$$I : H_0^1(\Omega) \to \mathbb{R},\ u \mapsto \int_\Omega \left(|\nabla u|^2 + |u|^2 - \alpha \cos(u)\sqrt{1 + |u|^2} \right) dx.$$

Show that I has a minimizer on the set

$$M := \{v \in H_0^1(\Omega) : \int_\Omega (|v|^2 + |v|)\,dx = 1\}.$$

Exercise 36:
Let $1 < p < \infty$ if $n = 1, 2$ and $1 < p < \frac{n+2}{n-2}$ if $n \geq 3$.

(a) Consider the nonlinear boundary value problem

$$\begin{cases} -\Delta u = |u|^{p-1}u, & \text{in } \Omega; \\ u = 0, & \text{auf } \partial\Omega. \end{cases}$$

Define a functional $I : H_0^1(\Omega) \to \mathbb{R}$ and an appropriate constraint $J(u) = 0$, such that the minimization problem of I under this constraint yields a solution to the problem.

(b) Consider now the problem

$$\begin{cases} -\,div(A(x)\nabla u) + V(x)u = \Gamma(x)|u|^{p-1}u, & \text{in } \Omega; \\ u = 0, & \text{auf } \partial\Omega. \end{cases}$$

where $A \in L^\infty(\mathbb{R}^n)^{n\times n}$ is some uniformly positive definit matrix and $V, \Gamma \in L^\infty(\mathbb{R}^n)$ satisfies $V, \Gamma \geq 0$ almost everywhere and $\Gamma \not\equiv 0$. How should the functional and the constraint look like in order to get a nontrivial solution?