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Nonlinear Boundary Value Problems

Exercise sheet 12

In the following exercises let $\Omega \subset \mathbb{R}^n$ be a bounded domain.

Exercise 37:
Let $K \subset H_0^1(\Omega)$ be a closed, convex subset. Moreover let $A \in L^\infty(\Omega)^{n\times n}$ be a uniformly positive definite matrix and $c \in L^\infty(\Omega)$ satisfy $c(x) \geq c_0 > 0$ almost everywhere and let $f \in L^2(\Omega)$ be given. Consider the functional

$$I : H_0^1(\Omega) \to \mathbb{R},\, u \mapsto \int_\Omega \left[\frac{1}{2}(\nabla u)^T A(x)\nabla u + \frac{1}{2}c(x)u^2 + f(x)u\right] dx.$$

(a) Find some constants $B, C > 0$ such that $I(u) \geq B\|u\|_{H_0^1(\Omega)} - C$ holds for all $u \in H_0^1(\Omega)$. Furthermore, prove that I has a unique minimizer u^* on K.

(b) Find a variational inequality for u^* in the spirit of Theorem X.4 from the lecture.

(c) Give some Radius $R > 0$ depending on B, C with the following property:

$$\{v \in H_0^1(\Omega) : \|v\|_{H_0^1(\Omega)} < R\} \subseteq K \text{ implies } u^* \in \mathring{K}$$

In the affirmative case, find an improved variational inequality for u^*.

Exercise 38:
Let $f \in L^2(\Omega)$. Show that the functional

$$I : V \to \mathbb{R},\, I(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - f(x)u\right] dx$$

has a unique minimzier u^* on $V := \{v \in H_0^1(\Omega) : |\nabla v| \leq 1 \text{ almost everywhere in } \Omega\}$. Moreover, prove that u^* satifies the following variational inequality:

$$\int_\Omega \nabla u^* \cdot \nabla(\phi - u^*)\, dx \geq \int_\Omega f(x)(\phi - u^*)\, dx \qquad (\phi \in V)$$