Nonlinear Boundary Value Problems

Exercise sheet 13

Exercise 39:

Check if the following mappings satisfy the Palais-Smale condition

(a) $f : \mathbb{R} \to \mathbb{R}, x \mapsto x \sin(x)$

(b) $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto \sin(x) + xy^2$

(c) $I : L^2((-1, 1)) \to \mathbb{R}, u \mapsto \int_{-1}^{1} u(t)^2 \, dt$

(d) $I : H^1_0((-1, 1)) \to \mathbb{R}, u \mapsto \int_{-1}^{1} u'(t)^2 \, dt + \int_{-1}^{1} u(t)^2 \, dt$

(e) $I : H^1_0((-1, 1)) \to \mathbb{R}, u \mapsto \int_{-1}^{1} u'(t)^2 \, dt + \int_{0}^{1} u(t)^2 \, dt$

(f) $I : H^1_0(\Omega) \to \mathbb{R}, u \mapsto \int_{\Omega} \left[ \frac{1}{2} |\nabla u|^2 - \frac{1}{4} |u|^4 \right] \, dx$, where $\Omega \subseteq \mathbb{R}^2$ is a bounded domain.

Exercise 40:

(a) Let $f \in C^1(\mathbb{R}^n, \mathbb{R})$, Show that $f$ satisfies the Palais-Smale condition provided that $|f| + |f'|$ is coercive.

(b) Let $f \in C^1(\mathbb{R}^n, \mathbb{R})$ be bounded from below and satisfy the Palais-Smale condition. Show that $f$ is coercive and attains its minimum.

Exercise 41:

Let $(X, \| \cdot \|)$ be a normed space and let $L : X \to \mathbb{R}$ be a continuously Fréchet-differentiable and convex (i.e. $L[(1-t)x + ty] \leq (1-t)L[x] + tL[y]$ for $x, y \in X, t \in [0, 1]$) mapping. Prove the following statements:

(a) $L[x] - L[y] \leq L[x](x - y)$ for $x, y \in X$

(b) If $L$ is strictly convex, then $L$ has at most one critical value.

(c) Give an example of a strictly convex functional without critical points.