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Nonlinear Boundary Value Problems

Exercise sheet 2

Exercise 5:
Let H be a Hilbert space. Prove the following basic properties of weak convergence:

(a) $u_n \to u$ in norm implies $u_n \rightharpoonup u$.

(b) If $u_n \rightharpoonup u$, then $(u_n)_{n=1}^\infty$ is bounded in H and

$$\|u\| \leq \liminf_{n\to\infty} \|u_n\|.$$

Hint: Use the principle of uniform boundedness.

(c) $u_n \rightharpoonup u$, $\|u_n\| \to \|u\|$ implies $u_n \to u$ in norm.

(d) If $\langle u_n, v\rangle$ converges for every $v \in H$, then there exists a $u \in H$, such that $u_n \rightharpoonup u$.

(e) Suppose that $(u_n)_{n=1}^\infty$ is bounded and that $\langle u_n, v\rangle \to \langle u, v\rangle$ for every $v \in D$, where D is a dense subset of H. Prove that $u_n \rightharpoonup u$.

(f) Use b) and e) to show: $u_n \rightharpoonup u \in H_0^1(\Omega)$ implies $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\Omega)^n$.

Exercise 6:
Let the monotone vector field a satisfy the conditions i)-iii) from the lecture. Consider the following boundary value problem on Ω:

$$A(u,\psi) := \int_\Omega [a(\nabla u)\cdot\nabla\psi + \alpha\sum_{i=1}^n b_i(u)(\partial_i u)\psi + g(u)\psi]\,dx = \int_\Omega f\psi\,dx \quad (\psi \in H_0^1(\Omega)) \quad (1)$$

for given real parameter $\alpha \neq 0$ and bounded continuous functions b_i. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is continuous and has the property that $g(u) \in L^2(\Omega)$ for all $u \in H_0^1(\Omega)$. Furthermore, let g satisfy $g(u)u \geq \beta$ $(u \in \mathbb{R})$ for some $\beta \in \mathbb{R}$. Show that if α is small enough, then the following *coercivity condition* holds:

$$A(u,u) \geq \delta_0\|u\|^2_{H_0^1(\Omega)} - \mu_0 \qquad (u \in H_0^1(\Omega)),$$

where $\delta_0 \in \mathbb{R}$, $\mu_0 \in \mathbb{R}$.

Exercise 7:
Assume that the assumptions from exercise 6 hold and α is small enough, such that the coercivity condition is satisfied. Extend the results of the Browder-Minty method from the lecture by proving the existence of a solution $u \in H_0^1(\Omega)$ for the boundary value problem (1). Proceed as follows:

— please turn around —

(a) Prove the existence of a Galerkin approximation $u_k \in V_k := span\{w_1, \cdots, w_k\}$, where $\{w_k\}_{k=1}^{\infty}$ is an orthonormal basis of $H_0^1(\Omega)$, as used in the lecture.

(b) Use the previous exercise to prove the existence of a weakly converging subsequence of $(u_k)_{k=1}^{\infty}$.

(c) Prove that the weak limit $u \in H_0^1(\Omega)$ is a solution of problem (1). In particular show the existence of some $\xi \in (L^2(\Omega))^n$ and some subsequence $(v_j)_{k=1}^{\infty} := (u_{k_j})_{j=1}^{\infty}$ with the following properties:

  (i) $\nabla v_j \rightharpoonup \xi$ in $L^2(\Omega)^n$

  (ii) $\int_\Omega [\xi \cdot \nabla \psi + \alpha \sum_{i=1}^{n} b_i(u)(\partial_i u)\psi + g(u)\psi] \, dx = \int_\Omega f \psi \, dx$ for all $\psi \in H_0^1(\Omega)$

  (iii) $\int_\Omega a(\nabla v_j) \cdot \nabla v_j \, dx \to \int_\Omega \xi \cdot \nabla u \, dx$

  (iv) $\int_\Omega (\xi - a(\nabla \psi))(\nabla u - \nabla \psi) \, dx \geq 0$ for all $\psi \in H_0^1(\Omega)$.