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WiSe 2016/17
04.11.2016

Nonlinear Boundary Value Problems

Exercise sheet 3

Exercise 8:
Let $\Omega \subset \mathbb{R}^n$ be open, bounded with Lipschitz boundary $\partial\Omega$. Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy the following Lipschitz condition:

$$|f(x, z_1, p_1) - f(x, z_2, p_2)| \leq L_1|z_1 - z_2| + L_2|p_1 - p_2| \qquad (x \in \Omega,\ z_1, z_2 \in \mathbb{R},\ p_1, p_2 \in \mathbb{R}^n)$$

Assume moreover that $f(\cdot, 0, 0) \in L^2(\Omega)$. Give some (reasonable) sufficient conditions on L_1, L_2, such that the boundary value problem

$$\begin{cases} -\Delta u = f(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution $u \in H_0^1(\Omega)$.

Exercise 9:
Let $A \in (L^\infty(\Omega))^{n\times n}$. We call a differential expression $L[u] := -\,div(A \cdot \nabla u)$ elliptic, if the matrix-valued function A is uniformly elliptic, i.e. there is a $c > 0$, such that:

$$\xi^T A(x)\xi \geq c|\xi|^2 \qquad \text{for almost all } x \in \Omega, \xi \in \mathbb{R}^n.$$

(a) Define the operator $H_0^1(\Omega) \to H^{-1}(\Omega)$, $u \mapsto L[u]$.

(b) Let Ω be bounded and let $H_0^1(\Omega)$ be the Hilbert space equipped with the inner product

$$\langle u, v\rangle_{H_0^1(\Omega)} := \langle \nabla u, \nabla v\rangle_{L^2(\Omega)}.$$

Show that $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isometric isomorphism.

(c) Show that for any non-negative function $c \in L^\infty(\Omega)$ the elliptic operator defined by

$$H_0^1(\Omega) \to H^{-1}(\Omega),\ u \mapsto L[u] + cu$$

is an isomorphism.

Exercise 10:
Define $H^{-1}(\mathbb{R})$ to be the Hilbert space of all continuous linear functionals $H^1(\mathbb{R}) \to \mathbb{R}$ as in the lecture. Define $\delta : H^1(\mathbb{R}) \to \mathbb{R}$, $u \mapsto u(0)$. Show that $\delta \in H^{-1}(\mathbb{R})$ and find an Element $v \in H^1(\mathbb{R})$ such that

$$\delta[u] = \langle u, v\rangle \qquad (u \in H_0^1(\mathbb{R}))$$

holds.

Hint: Use the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$ (cf. Adams - Sobolev Spaces).