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Nonlinear Boundary Value Problems

Exercise sheet 4

Exercise 11:
Consider the parabolic problem studied in the lecture on $[0, t_0]$. Prove that the function u glued together from the "local" solutions u_k on $[(k-1)t_1, kt_1]$ is a solution on $[0, t_0]$.

Hint: Derive first the integration-by-parts formula

$$\int_0^T \int_\Omega u(x,t) \frac{\partial \varphi(x,t)}{\partial t}\, dx\, dt = -\int_0^T \frac{\partial u}{\partial t}(t)[\varphi(\cdot, t)]\, dt + \int_\Omega u(x,T)\varphi(x,T)\, dx - \int_\Omega u(x,0)\varphi(x,0)\, dx$$

valid for $u \in L^2((0,T), H_0^1(\Omega)) \cap H^1((0,T), H^{-1}(\Omega))$ and $\varphi \in C^1([0,T], H_0^1(\Omega))$.

Exercise 12:
Let L be a second-order linear elliptic differential operator on $\Omega \subset \mathbb{R}^n$ (open and bounded, $\partial\Omega$ Lipschitz). Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy the growth condition $|f(u)| \leq C(1+|u|)$. Assume moreover that the problem

$$\begin{cases} L[u] = r & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has only the trivial solution for $r \equiv 0$ and hence, by the Fredholm Alternative Theorem, has a unique solution for every given $r \in L^2(\Omega)$. Give a sufficient condition on $\alpha > 0$, such that the boundary value problem

$$\begin{cases} L[u] = \alpha f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least one solution.

Hint: Rewrite the problem as a fixed point equation using the theory of weak solutions of elliptic differential equations and apply Schauder's fixed point theorem.

Exercise 13:
Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Moreover, let $A : \Omega \times \mathbb{R} \to \mathbb{R}^{n \times n}$ be a symmetric matrix valued function with the following properties:

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- There are constants $0 < a_0 \leq a_1$, such that:

$$a_0|\xi|^2 \leq \xi^T A(x,y)\xi \leq a_1|\xi|^2 \qquad (x \in \Omega,\, y \in \mathbb{R},\, \xi \in \mathbb{R}^n)$$

- $A(\cdot, y)$ is measurable for any $y \in \mathbb{R}$, $A(x, \cdot)$ is continuous for almost all $x \in \Omega$.

Furthermore, assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and there is a $C > 0$ such that $|f(z)| \leq C(1 + |z|)$ for any $z \in \mathbb{R}$, where $CC_P^2 < a_0$ holds (C_P denotes the Poincaré constant of Ω). Proof that the followin *quasilinear* boundary value problem

$$\begin{cases} -\,div(A(\cdot, u)\nabla u) = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a weak solution $u \in H_0^1(\Omega)$.