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Nonlinear Boundary Value Problems

Exercise sheet 5

Exercise 14:
Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $u \in H^1(\Omega)$. Prove that

$$u^+ = \max\{u, 0\},\ u^- = -\min\{u, 0\}$$

are in $H^1(\Omega)$. Moreover show that $u \in H^1(\Omega)$ with $u \leq 0$ almost everywhere on $\partial\Omega$ (w.r.t the surface measure on $\partial\Omega$) implies: $u^+ \in H_0^1(\Omega)$.

Exercise 15:
Consider on $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ the boundary value problem

$$\begin{cases} -\Delta u &= 1 + u^2 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases} \tag{1}$$

and show the existence of a solution $u \in H_0^1(\Omega)$ by constructing a suitable super- and subsolution in $H_0^1(\Omega)$.

Exercise 16:
Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth, bounded function such that $f(0) = 0,\ f'(0) > \lambda_1$ where λ_1 is the first eigenvalue of the Dirichlet Laplacian on Ω. Use the method of super- and subsolutions to show the existence of a solution to

$$\begin{cases} -\Delta u &= f(u) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases} \tag{2}$$

satisfying $u > 0$ on Ω. You may use without proof the fact that there is a bounded eigenfunction $w \in H_0^1(\Omega)$ with $w > 0$ corresponding to λ_1.

Exercise 17:
Show that the following boundary problems have a solution by constructing a suitable set of super- and subsolutions:

(a) For $\mu > \pi^2$:

$$\begin{cases} -u'' - \mu u + u^3 &= 0 \text{ on } (0,1) \\ u(0) = u(1) &= 0 \end{cases}$$

(b)

$$\begin{cases} -\varepsilon u'' - uu' + u &= 0 \text{ on } (0,1) \\ u(0) = 1,\ u(1) &= -1 \end{cases}$$

Hint: Choose $\overline{u}(x) = e^{-\mu x},\ \underline{u}(x) = -e^{\mu(x-1)}$ for some suitable μ.