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Nonlinear Boundary Value Problems

Exercise sheet 6

Exercise 18:
Let Ω be a bounded domain in $\mathbb{R}^n$. Assume that we have a sufficiently smooth solution $u:\,[0,T)\to H_0^1(\Omega)$ of the problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u &= f(u) \text{ in } \Omega \\ \qquad\quad u &= 0 \ \text{ on } \partial\Omega\times[0,T) \end{cases}$$

such that, for some $\beta>0$ and some $t^*\in(0,T]$, $\int_0^t \|u(\cdot,s)\|_\infty^\beta\,ds\to\infty$ for $t\to t^*-0$. Suppose that f suffices the following growth condition

$$|f(u)|\le C|u|^{\beta+1} \qquad (u\in\mathbb{R}).$$

Prove that we have the following lower bound on the blow-up time t^*:

$$t^*\ge (C\beta\|u(\cdot,0)\|_\infty^\beta)^{-1}$$

Hint: Consider the function $\psi_p(t)=\int_\Omega u(x,t)^{2p}\,dx$ for any $p\in[1,\infty)$ and derive the differential inequality

$$\psi_p'(t)\le 2Cp\|u(\cdot,t)\|_\infty^\beta\psi_p(t)$$

from which we obtain the estimate $\psi_p(t)\le\exp(2Cp\int_0^t\|u(\cdot,\tau)\|_\infty^\beta\,d\tau)\psi_p(0)$. By sending $p\to\infty$ deduce that

$$\|u(\cdot,t)\|_\infty\le\exp(2Cp\int_0^t\|u(\cdot,\tau)\|_\infty^\beta\,d\tau)\|u(\cdot,0)\|_\infty.$$

Exercise 19:
Let $f:\,\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ be a C^1-function and $u\in C^2(\mathbb{R}^n)$ be a classical solution of

$$-\Delta u+f(x,u)=0 \text{ on } \mathbb{R}^n$$

such that $\nabla u\in L^2(\mathbb{R}^n)$, $F(\cdot,u(\cdot))$, $F_1(\cdot,u(\cdot))\in L^1(\mathbb{R}^n)$, where

$$F(x,w):=\int_0^w f(x,t)\,dt$$
$$F_1(x,w):=\sum_{i=1}^n x_i\frac{\partial F(x,w)}{\partial x_i}.$$

— please turn over —

Prove the following version of the *Pohozaev* identity:

$$\frac{n-2}{2}\int_{\mathbb{R}^n}|\nabla u|^2\,dx+\int_{\mathbb{R}^n}\left(nF(x,u(x))+F_1(x,u(x))\right)\,dx=0$$

Hint: Use $x\cdot\nabla u$ as a test function and integrate the resulting expression first over a ball. Note that for any $g\in L^1(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n}g(x)\,dx=\int_0^\infty\int_{\{|x|=r\}}g(x)\,d\sigma dr$$

where σ denotes the surface measure on the ball.

Exercise 20:
Show that there is no nontrivial classical solution $u\in C^2(\mathbb{R}^3)$ to the equation

$$-\Delta u+\lambda_1|u|^{p-2}u+\lambda_2|u|^{q-2}u=0\text{ in }\mathbb{R}^3$$

with the properties $|\nabla u|\in L^2(\mathbb{R}^3)$, $F(\cdot,u(\cdot))$, $F_1(\cdot,u(\cdot))\in L^1(\mathbb{R}^3)$, where F, F_1 are defined as in exercise 19 with $f(x,u):=\lambda_1|u|^{p-2}u+\lambda_2|u|^{q-2}u$ and $1\leq p\leq q$ and λ_1, λ_2 satisfy the condition

$$-\lambda_1(p-6)\cdot\lambda_2(q-6)<0.$$

Hint: Test the equation with $\frac{1}{2}u$ and apply the identity from exercise 19.