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Nonlinear Boundary Value Problems

Exercise sheet 7

Exercise 21:
Let $\Omega \subset \mathbb{R}^n$ be an open domain and

$$Lu = -\sum_{i,j=1}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b \cdot \nabla u + cu$$

be a uniformly strongly elliptic differential operator with continuous coefficients. Suppose furthermore $c \geq 0$.

(a) Prove *Hopf's Lemma*: If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies $Lu \leq 0$ and, for some $x_0 \in \partial\Omega$ such that Ω satisfies the interior ball condition at x_0, one has

$$u(x_0) > u(x) \ (x \in \Omega), \qquad u(x_0) \geq 0,$$

then $\frac{\partial u}{\partial \nu}(x_0) > 0$.

Hint: For an interior ball $B \subset \Omega$ with $x_0 \in \partial\Omega$ (we may assume after possibly a change of coordinates, that $B = B_r(0) = \{x \in \mathbb{R}^n : |x| = r\}$) show that the function

$$v : B(0,r) \to \mathbb{R}, \, v(x) := e^{-\lambda|x|^2} - e^{-\lambda r^2}$$

satisfies $Lv \leq 0$ for some sufficiently large $\lambda > 0$. Then choose $\varepsilon > 0$ small enough and apply the weak maximum principle ($Lw \leq 0 \Rightarrow \ w \leq \max_{\partial\Omega} w$ for $w \in C^2(\Omega) \cap C^(\overline{\Omega})$) to the function $u + \varepsilon v - u(x_0)$ in the annulus $B_r(0) \setminus \overline{B_{r/2}(0)}$.

(b) Use a) to prove the strong maximum principle: If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$Lu \leq 0 \text{ on } \Omega$$

and u attains a nonnegative maximum over $\overline{\Omega}$ at an interior point, then u is constant on Ω.

(c) Prove the following consequence of a) and b): if $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies

$$Lu \geq 0 \text{ on } \Omega \text{ and } u \geq 0 \text{ on } \partial\Omega,$$

then either $u = 0$ of $u > 0$ in Ω. In the second case, we have in addition, for each $x_0 \in \partial\Omega$ such that Ω satisfies an interior ball condition at x_0,

$$u(x_0) = 0 \implies \frac{\partial u}{\partial \nu}(x_0) < 0.$$

— please turn over —

Exercise 22:
Let Ω be a domain with Lipschitz boundary and $2^* = \frac{2n}{n-2}$ be the critcal Sobolev exponent in the case $n \geq 3$.

Show that for $2 < p \leq 2^*$, if $n \geq 3$, $2 < p < \infty$, if $n = 1, 2$, the functional

$$I_p : H^1(\Omega) \to \mathbb{R},\ I_p(u) = \int_\Omega |u(x)|^p \, dx$$

is Fréchet-differentiable and compute its derivative.

Exercise 23:
Let X, Y, Z be Banach spaces, $G : X \to Y$ be differentiable in x_0 and $F : Y \to Z$ be differentiable in $y_0 = G(x_0)$.

(a) Show that there is a $\delta > 0$, such that for all $h \in X$ with $\|h\|_X \leq \delta$ one has

$$\|G(x_0 + h) - G(x)\|_Y \leq (1 + \|G'(x_0)\|_{X \to Y})\|h\|_X$$

(b) Show that $F \circ G$ is differentiable in x_0 and one has $(F \circ G)'(x_0) = F'(y_0) \cdot G'(x_0)$.