Nonlinear Boundary Value Problems

Exercise sheet 7

Exercise 21:
Let \( \Omega \subset \mathbb{R}^n \) be an open domain and
\[
Lu = -\sum_{i,j=1}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b \cdot \nabla u + cu
\]
be a uniformly strongly elliptic differential operator with continuous coefficients. Suppose furthermore \( c \geq 0 \).

(a) Prove Hopf’s Lemma: If \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) satisfies \( Lu \leq 0 \) and, for some \( x_0 \in \partial\Omega \) such that \( \Omega \) satisfies the interior ball condition at \( x_0 \), one has
\[
u(x_0) > u(x) (x \in \Omega), \quad u(x_0) \geq 0,
\]
then \( \frac{\partial u}{\partial \nu}(x_0) > 0 \).

Hint: For an interior ball \( B \subset \Omega \) with \( x_0 \in \partial\Omega \) (we may assume after possibly a change of coordinates, that \( B = B_r(0) = \{ x \in \mathbb{R}^n : |x| = r \} \)) show that the function
\[
v : B(0, r) \to \mathbb{R}, \quad v(x) := e^{-\lambda|x|^2} - e^{-\lambda r^2}
\]
satisfies \( Lv \leq 0 \) for some sufficiently large \( \lambda > 0 \). Then choose \( \varepsilon > 0 \) small enough and apply the weak maximum principle (\( Lw \leq 0 \Rightarrow w \leq \max_{\partial\Omega} w \) for \( w \in C^2(\Omega) \cap C^1(\overline{\Omega}) \)) to the function \( u + \varepsilon v - u(x_0) \) in the annulus \( B_r(0) \setminus B_r/2(0) \).

(b) Use a) to prove the strong maximum principle: If \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) satisfies
\[Lu \leq 0 \text{ on } \Omega\]
and \( u \) attains a nonnegative maximum over \( \overline{\Omega} \) at an interior point, then \( u \) is constant on \( \Omega \).

(c) Prove the following consequence of a) and b): if \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) satisfies
\[Lu \geq 0 \text{ on } \Omega \text{ and } u \geq 0 \text{ on } \partial\Omega,
\]
then either \( u = 0 \) or \( u > 0 \) in \( \Omega \). In the second case, we have in addition, for each \( x_0 \in \partial\Omega \) such that \( \Omega \) satisfies an interior ball condition at \( x_0 \),
\[
u(x_0) = 0 \implies \frac{\partial u}{\partial \nu}(x_0) < 0.
\]
Exercise 22:
Let \( \Omega \) be a domain with Lipschitz boundary and \( 2^* = \frac{2n}{n-2} \) be the critical Sobolev exponent in the case \( n \geq 3 \).

Show that for \( 2 < p \leq 2^* \), if \( n \geq 3 \), \( 2 < p < \infty \), if \( n = 1, 2 \), the functional

\[
I_p : H^1(\Omega) \to \mathbb{R}, \quad I_p(u) = \int_{\Omega} |u(x)|^p \, dx
\]

is Fréchet-differentiable and compute its derivative.

Exercise 23:
Let \( X,Y,Z \) be Banach spaces, \( G : X \to Y \) be differentiable in \( x_0 \) and \( F : Y \to Z \) be differentiable in \( y_0 = G(x_0) \).

(a) Show that there is a \( \delta > 0 \), such that for all \( h \in X \) with \( \|h\|_X \leq \delta \) one has

\[
\|G(x_0 + h) - G(x)\|_Y \leq (1 + \|G'(x_0)\|_{X \to Y}) \|h\|_X
\]

(b) Show that \( F \circ G \) is differentiable in \( x_0 \) and one has \( (F \circ G)'(x_0) = F'(y_0) \cdot G'(x_0) \).