Nonlinear Boundary Value Problems

Exercise sheet 8

Exercise 24:
Show that the boundary value problem
\[
\begin{aligned}
-\text{div}(A\nabla u) + b \cdot \nabla u + f(u, \cdot) & = 0 \text{ in } \Omega \\
u & = g \text{ on } \partial \Omega
\end{aligned}
\]
can be written in variational form by introducing a new unknown \( w \) by \( u = e^{\psi(\cdot)}w \) for some \( \psi \), provided that \( A^{-1}(\cdot)b(\cdot) \) is a gradient on \( \Omega \).

Exercise 25:
Consider a chain with homogeneously distributed mass attached to the points \((-a, b)\) and \((a, b)\) with \(a, b > 0\). The corresponding variational problem, i.e. the minimization of the potential energy, reads:

Minimize \( \int_{-a}^{a} y(x) \sqrt{1 + y'(x)^2} \, dx \),

where the chain is parametrized by \((x, y(x))\), \((x \in (-a, a))\).

(a) Show that for a solution \( y_0(x) \) of the corresponding Euler-Lagrange equation one has
\[
H(x) = y_0'(x) \frac{\partial L}{\partial p}(y_0'(x), y_0(x)) - L(y_0'(x), y_0(x)) = c
\]
for some \( c \in \mathbb{R} \).

(b) Use (a) to find a solution of the Euler-Lagrange equation.

Exercise 26:
Let \( \Omega \subset \mathbb{R}^n \) be a domain. Find the corresponding Euler-Lagrange equations to the following functionals on \( H^2_0(\Omega) \):

(a) \( J[u] = \int_{\Omega} \left[ \frac{(\Delta u)^2}{2} - F(x, u) \right] \, dx \), where \( F(x, t) := \int_{0}^{t} f(x, s) \, ds \)

(b) \( J[u] = \int_{\Omega} ((\Delta u)^2 - |u_{xx}|^2)u \, dx \)

Note that in contrast to the situation in the lecture the Lagrange function depends also on second derivatives of \( u \).