Karlsruher Institut für Technologie (KIT)
Institut für Analysis
Prof. Dr. Michael Plum
M. Sc. Peter Rupp

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Nonlinear Boundary Value Problems

Exercise sheet 8

Exercise 24:
Show that the boundary value problem

$$\begin{cases} -div(A\nabla u) + b\cdot\nabla u + f(u,\cdot) & = 0 \text{ in } \Omega \\ u & = g \text{ on } \partial\Omega \end{cases}$$

can be written in variational form by introducing a new unknown w by $u = e^{\psi(\cdot)}w$ for some ψ, provided that $A^{-1}(\cdot)b(\cdot)$ is a gradient on Ω

Exercise 25:
Consider a chain with homogeneously distributed mass attached to the points $(-a, b)$ and (a, b) with $a, b > 0$. The corresponding variational problem, i.e. the minimization of the potential energy, reads:

$$\text{Minimize } \int_{-a}^{a} y(x)\sqrt{1+y'(x)^2}\,dx,$$

where the chain is parametrized by $(x, y(x))$, $(x \in (-a, a))$.

(a) Show that for a solution $y_0(x)$ of the corresponding Euler-Lagrange equation one has

$$H(x) = y_0'(x)\frac{\partial L}{\partial p}(y_0'(x), y_0(x)) - L(y_0'(x), y_0(x)) = c$$

for some $c \in \mathbb{R}$.

(b) Use (a) to find a solution of the Euler-Lagrange equation.

Exercise 26:
Let $\Omega \subset \mathbb{R}^n$ be a domain. Find the corresponding Euler-Lagrange equations to the following functionals on $H_0^2(\Omega)$:

(a) $J[u] = \int_\Omega \left[\frac{(\Delta u)^2}{2} - F(x,u)\right] dx,$ where $F(x,t) := \int_0^t f(x,s)\,ds$

(b) $J[u] = \int_\Omega ((\Delta u)^2 - |u_{xx}|_2^2)u\,dx$

Note that in contrast to the situation in the lecture the Lagrange function depends also on second derivatives of u.