Karlsruher Institut für Technologie (KIT)
Institut für Analysis
Prof. Dr. Michael Plum
M. Sc. Peter Rupp

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Nonlinear Boundary Value Problems

Exercise sheet 9

Exercise 28:
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $h \in C(\partial\Omega)$. Consider the nonlinear boundary value problem

$$\begin{cases} -\Delta u + \frac{n}{2}\frac{|\nabla u|^2}{u} = 0 & \text{in } \Omega \\ u = h & \text{on } \partial\Omega \end{cases}$$

Calculate a corresponding Lagrangian and write the problem in a variational form, that is, define a functional I on an appropriate space V such that the given equation is the corresponding Euler-Lagrange equation.

Hint: For the Lagrangian multiply the equation by $2u^{-n}$

Exercise 29:
Let $\Omega \subset \mathbb{R}^n$ be a nonempty, bounded domain, $\lambda > 0$ and $I : H_0^1(\Omega) \to \mathbb{R}$ be defined by

$$I(u) = \int_\Omega \left[\frac{|\nabla u|^2}{2} + \lambda \cos(u) \right] dx \qquad (u \in H_0^1(\Omega))$$

Prove the following statements:

(a) I is weakly lower semicontinuous and there exists a minimizer $u_0 \in H_0^1(\Omega)$.

(b) There is $\lambda_* > 0$, such that $u_0 = 0$ for all $0 < \lambda < \lambda_*$.

(c) There is $\lambda^* > 0$, such that $u_0 \neq 0$ for all $\lambda > \lambda^*$.

(d) The minimizer u_0 is a weak solution of the corresponding Euler-Lagrange equation.

Hint: For (b) and (c) use the Poincaré inequality.

Exercise 30:
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\alpha \in \mathbb{R}$. Show that the functional $I : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$I(u) = \int_\Omega \left[\frac{1}{2}(|\nabla u|^2 + |u|^2) + \alpha\sqrt{|u|^2 + 1} - u \right] dx \qquad (u \in H_0^1(\Omega)),$$

has a minimizer $u_0 \in H_0^1(\Omega)$. Calculate the corresponding Euler-Lagrange equation and show that u_0 is a weak solution of the Euler-Lagrange equation.