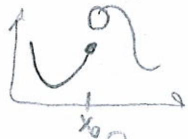
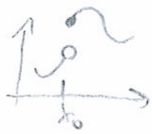
(X, ϱ) a METRIC SPACE 24.05.2016

$f : X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ IS CALLED LOWER SEMI CONTINUOUS (UPPER SEMI CONT- at point $x_0 \in X$ IFF

$$\liminf_{x \xrightarrow{\varrho} x_0} f(x) \geq f(x_0) \qquad \left(\limsup_{x \to x_0} f(x) \leq f(x_0) \right)$$





- IF f IS LSC & USC $\Rightarrow$ f IS CONTINUOUS
- f IS LSC $\Longleftrightarrow$ $-f$ IS USC
- f, g LSC, $\alpha, \beta \geq 0$ $\Longrightarrow$ $\alpha f + \beta g$ LSC $\{\mp\infty \pm\infty\}$
- f, g LSC, $f, g \geq 0$ $\Rightarrow$ fg IS LSC $\{0 \cdot \infty\}$
- f, g LSC $\Longrightarrow$ $\min\{f, g\}$, $\max\{f, g\}$ ARE LSC.

THM. f_n CONT $f_n \leq f_{n+1}$, $f_n \xrightarrow[\text{POINTWISE}]{} f$ $\Longrightarrow$ f IS LSC.

THM (BAIRE) EVERY LSC FUNCTION IS A POINTWISE LIMIT OF INCREASING SEQUENCE OF CONTINUOUS FUNCTIONS.

THM (BOLZANO-WEIERSTRASS).

(X, ϱ) COMPACT, $f : X \to \mathbb{R}$ LSC $\Rightarrow$

$$\bigvee_{x \in X} f(x) = \inf_{x \in X} f(x).$$

PROOF. $y = \inf_{x \in X} f(x)$.

THERE EXISTS $(x_n)_{n \in \mathbb{N}}$: $y = \lim_{n \to \infty} f(x_n)$

X COMPACT $\Rightarrow$ W.L.O.G. x_n CONVERGENT.

$$x_n \xrightarrow{X} x_0$$

$$\text{LSC} \Rightarrow \liminf_{n \to \infty} f(x_n) \geq f(x_0)$$

$$\overset{\parallel}{\inf_{x \in X} f(x)}$$

X NORMED SPACE

24. 05. 2016.

$f: X \to \mathbb{R}$ IS CALLED COERSIVE IFF FOR EVERY SEQUENCE $(x_n)_{n\in\mathbb{N}}$: $(f(x_n))_{n\in\mathbb{N}}$ BOUNDED $\Rightarrow$ $(\|x_n\|)_{n\in\mathbb{N}}$ BOUNDED

A SET $A \subseteq X$ IS CALLED SEQUENTIONALLY WEAKLY CLOSED IIF FOR EVERY SEQ $(a_n)_{n\in\mathbb{N}}$ OF ELEMENTS OF A

$a_n \rightharpoonup x$ FOR SOME $x \in X$, WE HAVE $x \in A$.

$f: X \to \mathbb{R}$ IS CALLED SEQUENTIONALLY WEAKLY LOWER SEMI CONTINUOUS IFF $\liminf\limits_{x \rightharpoonup x_0} f(x) \geq f(x_0)$

THM 1

V BE a REFLEXIVE SPACE $\{V^{**} = V\}$

$(V_n)_{n\in\mathbb{N}}$ a BOUNDED SEQUENCE

THEN (V_n) HAS a WEAKLY CONVERGENT SUBSEQUENCE.

THM 2.

V REFLEXIVE SPACE, $\emptyset \neq X \subseteq V$ IS SEQUENTIONALLY WEAKLY CLOSED, $f: X \to \mathbb{R}$ COERSIVE & SEQ WLSC.

THEN THERE EXISTS $x_0 \in X$: $f(x_0) = \inf\limits_{x\in X} f(x)$

PROOF. $y = \inf\limits_{x\in X} f(x)$

$(x_n)_{n\in\mathbb{N}}$. $f(x_n) \to y$.

$(f(x_n))_{n\in\mathbb{N}}$ BOUNDED $\xRightarrow{f \text{ COERSIVE}}$ $(\|x_n\|)_{n\in\mathbb{N}}$ BOUNDED $\xRightarrow{\text{THM 1}}$ (x_n) HAS a WEAKLY CONVERGENT SUBSEQUENCE.

W.L.O.G. $x_n \rightharpoonup x_0$ $\{x_0 \in X$ BY SEQ. WEAK CLOSEDNESS OF $X\}$.

$y = \liminf\limits_{n\to\infty} f(x_n) \geq f(x_0)$

$y = f(x_0)$.