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Aspects of nonlinear wave equations

Sheet 1

In this problem sheet we consider the two-dimensional wave equation in a non-homogeneous medium, i.e., the wave-speed $c = c(x_2)$ depends on the x_2 variable. In the first problem you can derive Snell's law of reflection/refraction and in the second problem you can discuss so-called guided modes in a wave-guide. For both problems the underlying equation is

$$\text{(LWE)} \qquad \frac{1}{c^2(x_2)} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^2 \times \mathbb{R}.$$

Problem 1 Consider two materials separated by the line $x_2 = 0$ and for $c_+, c_- > 0$ let

$$c(x_2) = \begin{cases} c_- & \text{for } x_2 < 0, \\ c_+ & \text{for } x_2 > 0. \end{cases}$$

Consider $u(x,t) = e^{i\omega t} v(x)$ with $\omega > 0$ and

$$v(x) = \begin{cases} Ae^{ik\cdot x} + A'e^{ik'\cdot x} & \text{for } x_2 < 0, \\ A''e^{ik''\cdot x} & \text{for } x_2 > 0, \end{cases}$$

$A, A', A'' \in \mathbb{R}$ and wave-vectors

$$k = \frac{\omega}{c_-}\begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix}, \quad k' = \frac{\omega}{c_-}\begin{pmatrix} \sin\phi' \\ -\cos\phi' \end{pmatrix}, \quad k'' = \frac{\omega}{c_+}\begin{pmatrix} \sin\phi'' \\ \cos\phi'' \end{pmatrix}, \qquad \phi, \phi', \phi'' \in \left[0, \frac{\pi}{2}\right)$$

(a) Verify that u satisfies (LWE) in $\mathbb{R}^2 \setminus \{x_2 = 0\} \times \mathbb{R}$.

(b) ~~$v \in C^1(\mathbb{R}^2)$ if and only if~~

$$\phi = \phi', \quad \frac{c_-}{c_+} = \frac{\sin\phi''}{\sin\phi}.$$

~~In this case, how does one choose A', A'' in relation to A?~~

Correction: $v \in C(\mathbb{R}^2)$ requires

$$\phi = \phi', \quad \frac{c_+}{c_-} = \frac{\sin\phi''}{\sin\phi}.$$

In this case, how does one need to choose A', A'' in relation to A to get $v \in C^1(\mathbb{R}^2)$?

(c) Under the conditions of (b), show that v is a weak solution of $\frac{-\omega^2}{c^2(x_2)} v - \Delta v = 0$ in $\mathbb{R}^2$, i.e.,

$$\int_{\mathbb{R}^2} -\frac{\omega^2}{c^2(x_2)} v\psi + \nabla v \cdot \nabla \psi \, dx = 0 \text{ for all } \psi \in C_c^\infty(\mathbb{R}^2).$$

Problem 2 For $c_1, c_2 > 0$ and $a > 0$ ($2a$=width of a wave-guide parallel to the x_1-axsis) let

$$c(x_2) = \begin{cases} c_1 & \text{for } -a < x_2 < a, \\ c_2 & \text{for } |x_2| > a. \end{cases}$$

For given frequency $\omega > 0$ we are looking for wave-numbers $k > 0$ such that

$$u(x,t) = e^{i(kx_1 - \omega t)} v(x_2)$$

solves (LWE) with $v(x_2) \to 0$ as $|x_2| \to \infty$. Such a solution is called a guided-mode.

(a) Show that v must solve

$$(*) \qquad \left(-\frac{\omega^2}{c(x_2)^2} + k^2\right) v - v'' = \text{ on } \mathbb{R}.$$

Consider from now on functions v that solve $(*)$ pointwise in $\mathbb{R} \setminus \{-a, a\}$, decay to 0 as $|x_2| \to \infty$ and belong to $C^1(\mathbb{R})$. Show that such solutions solve $(*)$ in the weak sense, cf. Problem 1(c).

(b) Show that necessarily $k^2 \in \left(\frac{\omega^2}{c_2^2}, \frac{\omega^2}{c_1^2}\right)$, and hence $c_1 < c_2$.

(c) Solve $(*)$ explicitely in $(-a, a)$ and $\mathbb{R} \setminus [-a, a]$ such that v is even, i.e., $v(x_2) = v(-x_2)$.

(d) Find the relation between the unknown k and the given values c_1, c_2, ω and a such that $v \in C^1(\mathbb{R})$.

(e) Explain that, for a small enough, the wave-guide supports (up to multiples) exactly one even guided-mode.