Karlsruhe Institute of Technology (KIT)
Institute for Analysis
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Summer Semester 2016
04.07.2016

Aspects of nonlinear wave equations

Sheet 10

Problem 1

We continue Problem 1 from the Sheet 9. The main new ingredient is (a) and (d). (b) and (c) can be done in the same way as in the lecture.

Let $D = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0,\pi)$. For $1 < q < \infty$ let $\mathcal{L}_q = \{\psi \in L^q(D) : \psi \text{ is even in } x \text{ and } t\}$. It is a Banach space with norm $\|\cdot\|_{L^q(D)}$. Functions in $\mathcal{L}_q$ are extended π-periodically in time. Likewise, $\mathcal{C}^{k,\alpha} = \left\{\psi \in C^{k,\alpha}([-\frac{\pi}{2}, \frac{\pi}{2}] \times \mathbb{R})\colon \psi \text{ is even in } x \text{ and } t, \pi\text{-periodic in } t\right\}$ is a Banach-space with $\|\cdot\|_{C^{k,\alpha}}$ for $k = 0, 1, 2$ and $\alpha \in [0, 1]$.

(a) Show that for $f \in \mathcal{C}^1$ the unique classical solution v of $(\star\star)$ is given by

$$v(x,t) = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{t+x-y}^{t-x+y} f(y,s)\, \mathrm{d}s\, \mathrm{d}y - \frac{1}{2}\int_{x}^{\frac{\pi}{2}} \int_{t+x-y}^{t-x+y} f(y,s)\, \mathrm{d}s\, \mathrm{d}y + c,$$

$$c = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{t-\frac{\pi}{2}-y}^{t+\frac{\pi}{2}+y} f(y,s)\, \mathrm{d}s\, \mathrm{d}y.$$

Check first that c is independent of t.

(b) Let us write $Kf := v$ with f in the spaces given next and v as in (a). Check that the following operators are continuous:

$$K\colon \mathcal{L}_q \to \mathcal{C}^{0,\alpha},\ \alpha = 1 - \frac{1}{q},$$
$$K\colon \mathcal{C}^{k,\alpha} \to \mathcal{C}^{k+1,\alpha},\ \alpha \in (0,1),\, k \in \{0,1\}.$$

(c) For $r = \frac{1}{p}$, $p > 1$ show that a solution $V \in \mathcal{L}_{r+1} \setminus \{0\}$ of $KV = -|V|^{r-1} V$ exists.

(d) With V from (c) show that $u := -|V|^{r-1} V$ is a classical solution of

$$\begin{cases} u_{tt} - u_{xx} = -|u|^{p-1} u & \left((x,t) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}\right), \\ u\left(-\frac{\pi}{2}, t\right) = u\left(\frac{\pi}{2}, t\right) = 0 & (t \in \mathbb{R}), \\ u(x, t+\pi) = u(x,t) & \left((x,t) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}\right), \\ u(-x,t) = u(x,t) = u(x,-t) & \left((x,t) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}\right). \end{cases}$$

Recall:
Oral exams dates: 08.09.2016 (Thursday), 07.10.2016 (Friday)