Aspects of nonlinear wave equations

Problem 1

We continue Problem 1 from the Sheet 9. The main new ingredient is (a) and (d). (b) and (c) can be done in the same way as in the lecture.

Let $D = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, \pi)$. For $1 < q < \infty$ let $L_q = \{ \psi \in L^q(D) : \psi \text{ is even in x and t} \}$. It is a Banach space with norm $\| \cdot \|_{L_q(D)}$. Functions in $L_q$ are extended $\pi$-periodically in time. Likewise, $C^{k,\alpha} = \{ \psi \in C^{k,\alpha}([-\frac{\pi}{2}, \frac{\pi}{2}] \times \mathbb{R}) : \psi \text{ is even in x and t, } \pi \text{-periodic in t} \}$ is a Banach-space with $\| \cdot \|_{C^{k,\alpha}}$ for $k = 0, 1, 2$ and $\alpha \in [0, 1]$.

(a) Show that for $f \in C^1$ the unique classical solution $v$ of (***) is given by

$$v(x, t) = \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{t-x-y}^{t-x+y} f(y, s) \, ds \, dy - \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{t+\pi-x-y}^{t+\pi-x+y} f(y, s) \, ds \, dy + c,$$

$$c = \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{t-\frac{\pi}{2}-y}^{t+\frac{\pi}{2}-y} f(y, s) \, ds \, dy.$$ Check first that $c$ is independent of $t$.

(b) Let us write $Kf := v$ with $f$ in the spaces given next and $v$ as in (a). Check that the following operators are continuous:

$$K : L_q \to C^{0,\alpha}, \alpha = 1 - \frac{1}{q},$$

$$K : C^{k,\alpha} \to C^{k+1,\alpha}, \alpha \in (0, 1), k \in \{0, 1\}.$$  

(c) For $r = \frac{1}{p}$, $p > 1$ show that a solution $V \in L_{r+1} \setminus \{0\}$ of $KV = -|V|^{r-1}V$ exists.

(d) With $V$ from (c) show that $u := -|V|^{r-1}V$ is a classical solution of

$$\begin{cases}
    u_{tt} - u_{xx} = -|u|^{r-1}u & \text{for } (x, t) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\
    u(-\frac{\pi}{2}, t) = u(\frac{\pi}{2}, t) = 0 & \text{for } t \in \mathbb{R}, \\
    u(x, t+\pi) = u(x, t) & \text{for } (x, t) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\
    u(-x, t) = u(x, t) = u(x, -t) & \text{for } (x, t) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}.
\end{cases}$$

Recall:
Oral exams dates: 08.09.2016 (Thursday), 07.10.2016 (Friday)