Karlsruhe Institute of Technology (KIT)
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Aspects of nonlinear wave equations

Sheet 11

Problem 1

Assume $m > 0$. In this problem we extend the results of Section 4.2 to standing waves of

$$\begin{cases} u_{tt} + u_{xxxx} + mu = \pm u^3 & ((x,t) \in (0,\pi) \times \mathbb{R}), \\ u(0,t) = u(\pi,t) = 0 & (t \in \mathbb{R}), \\ u_{xx}(0,t) = u_{xx}(\pi,t) = 0 & (t \in \mathbb{R}), \\ u(x,t+T) = u(x,t) & ((x,t) \in (0,\pi) \times \mathbb{R}), \\ u(x,t) = u(x,-t) & ((x,t) \in (0,\pi) \times \mathbb{R}). \end{cases}$$

Notice that now there are four boundary conditions – they are called Navier-conditions.

(a) Find the free vibrations of the operator $L = \partial_t^2 + \partial_x^4 + m$ under the above Navier- and periodicity conditions. *Hint:* first solve the eigenvalue problem $\psi'''' + m\psi = \lambda\psi$ on $(0,\pi)$ with $\psi(0) = \psi''(0) = \psi(\pi) = \psi''(\pi) = 0$. The frequencies $(\omega_l)_{l\in\mathbb{N}}$ of the free vibrations with $\omega_1 < \omega_2 < \ldots$ are related to the eigenvalues $(\lambda_l)_{l\in\mathbb{N}}$ with $\lambda_1 < \lambda_2 < \ldots$.

(b) For $\omega > 0$ consider the operator $L_\omega = \omega^2\partial_t^2 + \partial_x^4 + m$ with Navier-conditions at $x = 0, x = \pi$, 2π-periodicity and evenness in t. Fix a free-vibration frequency ω_{l_0} from (a) and assume $m \in \mathbb{R} \setminus \mathbb{Q}$. Show that $K_{l_0} := \operatorname{Ker} L_{\omega_{l_0}} = \operatorname{span}\{\psi_{l_0}(x)\cos t\}$.

(c) Let $m \in \mathbb{R}\setminus\mathbb{Q}$ and ω be such that $|\omega - \omega_1| \le \frac{\omega_1}{4}$ with ω_1 from (a). Find a γ-nonresonance condition for ω such that L_ω has a bounded inverse $L_\omega^{-1} : \mathcal{H} \cap K_1^\perp \to \mathcal{H} \cap K_1^\perp$ for a suitable Hilbert-space $\mathcal{H}$. Is this nonresonance condition covered by the Assumption of Section 4.2? *Hints:* $\perp$ is with respect to $L^2(D)$ for $D = (0,\pi) \times (0,2\pi)$. Define $\mathcal{H} \subset H^{4,2}(D)$ where $H^{4,2}(D)$ stands for functions v with $v_{xxxx}, v_{tt} \in L^2(D)$.

Recall:
Oral exams dates: 08.09.2016 (Thursday), 07.10.2016 (Friday)