Karlsruhe Institute of Technology (KIT)
Institute for Analysis
Prof. Dr. Wolfgang Reichel
Piotr Idzik, M.Sc.

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Aspects of nonlinear wave equations

Sheet 12

Problem 1

We use the notation from Sheet 9/Sheet 10. Assume $r \in \mathbb{N}$, $r \geq 2$, $\alpha, \epsilon \in \mathbb{R}$ and $h \in \mathcal{C}^{1,\alpha}(\overline{D})$ with $D = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, \pi)$. In this problem the idea is to use the implicit function theorem once again to find solutions of

$$\begin{cases} u_{tt} - u_{xx} &= \alpha u^r + \epsilon h(x,t) & \big((x,t) \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big) \times \mathbb{R}\big)\,, \\ u\big(-\frac{\pi}{2}, t\big) = u\big(\frac{\pi}{2}, t\big) &= 0 & (t \in \mathbb{R})\,, \\ u(x, t+\pi) &= u(x,t) & \big((x,t) \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big) \times \mathbb{R}\big)\,, \\ u(-x,t) = u(x,t) &= u(x,-t) & \big((x,t) \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big) \times \mathbb{R}\big)\,. \end{cases} \qquad (\star)$$

(a) With the help of the operator K from Sheet 9/Sheet 10 rewrite $(\star)$ as $u = K(\alpha u^r + \epsilon h)$ for $u \in \mathcal{C}$. Here $\mathcal{C} = \big\{\psi \in C([-\frac{\pi}{2}, \frac{\pi}{2}] \times \mathbb{R})\colon \psi \text{ is even in } x \text{ and } t, \pi\text{-periodic in } t\big\}$ is a Banach space with the norm $\|\cdot\|_\infty$.

(b) Show that $F : \mathcal{C} \times \mathbb{R} \mapsto \mathcal{C}$ defined by $u \mapsto u - K(\alpha u^r + \epsilon h)$ is a C^1-map and compute its derivatives $\partial_u F$, $\partial_\epsilon F$. Start with the simplest case $r = 2$.

(c) Show that there exists $\epsilon_0 > 0$ such that $(\star)$ has a classical solution $u = u_\epsilon$ for every $\epsilon \in (-\epsilon_0, \epsilon_0)$.

(d) Find conditions on $f : \overline{D} \times \mathbb{R} \to \mathbb{R}$ so that the same existence statement holds for

$$\begin{cases} u_{tt} - u_{xx} &= f(x,t,u) + \epsilon h(x,t) & \big((x,t) \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big) \times \mathbb{R}\big)\,, \\ u\big(-\frac{\pi}{2}, t\big) = u\big(\frac{\pi}{2}, t\big) &= 0 & (t \in \mathbb{R})\,, \\ u(x, t+\pi) &= u(x,t) & \big((x,t) \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big) \times \mathbb{R}\big)\,, \\ u(-x,t) = u(x,t) &= u(x,-t) & \big((x,t) \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big) \times \mathbb{R}\big)\,. \end{cases} \qquad (\star\star)$$

Recall:
Oral exams dates: 08.09.2016 (Thursday), 07.10.2016 (Friday)