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Aspects of nonlinear wave equations

Sheet 3

Problem 1 For $\underline{f \in C^2(\mathbb{R})}$, $\underline{f(0) = 0}$, $f'(0) = 1$ consider the fourth order wave equation

$$U_{tt} + U_{xxxx} + f(U) = 0. \tag{$\star$}$$

Traveling waves $U(x,t) = v(x - \omega t)$ satisfy

$$v^{\text{iv}} + \omega^2 v'' + f(v) = 0.$$

If we set $v(x) = \frac{4}{\omega^4} u\left(\frac{\omega}{\sqrt{2}} x\right)$, then u satisfies

$$u^{\text{iv}} + 2u'' + g(u) = 0 \text{ with } g(y) = f\left(\frac{4}{\omega^4} y\right). \tag{$\star\star$}$$

(a) Find a first integral for $(\star\star)$.

(b) Write $(\star\star)$ as a Hamiltonian system with

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -u' - u''' \\ u + u'' \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} u \\ u' \end{pmatrix}.$$

(c) Show that $(\star)$ has periodic traveling waves for $\omega \in \left(\sqrt{2}, \infty\right)$ by means of the Lyapunov center theorem.

Remark: The relevant matrix is $\begin{pmatrix} \frac{\partial^2 H}{\partial p \partial q}(p_0, q_0) & \frac{\partial^2 H}{\partial q^2}(p_0, q_0) \\ -\frac{\partial^2 H}{\partial p^2}(p_0, q_0) & -\frac{\partial^2 H}{\partial q \partial p}(p_0, q_0) \end{pmatrix}$. Note the minus sign in the lower right entry.

Problem 2 Consider $u_{tt} - u_{xx} = u\,(1 - u^2)$. Find the explicit form of traveling wave $u(x,t) = v(x - \omega t)$, $\omega \in (-1,1) \setminus \{0\}$ with a heteroclinic profile $v(0) = 0$, $\lim_{s \to \pm\infty} v(s) = \pm 1$.