Karlsruhe Institute of Technology (KIT)
Institute for Analysis
Prof. Dr. Wolfgang Reichel
Piotr Idzik, M.Sc.

Summer Semester 2016
18.5.2016

Aspects of nonlinear wave equations

Sheet 5

Problem 1 In this exercise we analyze the operator $L = \frac{\mathrm{d}^4}{\mathrm{d}x^4} + c^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1$.

(a) For $0 \le c^2 < 2$ construct a Green-function such that for every $f \in C(\mathbb{R}) \cap L^2(\mathbb{R})$ the unique solution $u \in C^4(\mathbb{R}) \cap L^2(\mathbb{R})$ of

$$Lu = f \text{ on } \mathbb{R} \qquad (\star)$$

is given by $u(x) = \int_{\mathbb{R}} G(x,y) f(y)\,\mathrm{d}y$, $x \in \mathbb{R}$. Proceed as follows: if $\widehat{u}$, $\widehat{f}$ is the Fourier transform of u, f given by

$$\widehat{u}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(\xi)\, e^{-\mathrm{i}k\xi}\,\mathrm{d}\xi,$$
$$u(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{u}(\xi)\, e^{\mathrm{i}k\xi}\,\mathrm{d}\xi,$$

then the solution of $(\star)$ is given by

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{k^4 - c^2 k^2 + 1} f(\xi)\, e^{\mathrm{i}k(x-\xi)}\,\mathrm{d}\xi\,\mathrm{d}k.$$

Decompose

$$\begin{aligned}\frac{1}{k^4 - c^2k^2 + 1} &= \frac{1}{\mu - \overline{\mu}} \left(\frac{1}{k^2 - \mu} - \frac{1}{k^2 - \overline{\mu}} \right) \\ &= \frac{1}{\mu - \overline{\mu}} \left(\frac{1}{2\lambda_1} \left(\frac{1}{k - \lambda_1} - \frac{1}{k + \lambda_1} \right) - \frac{1}{2\lambda_2} \left(\frac{1}{k - \lambda_2} - \frac{1}{k + \lambda_2} \right) \right),\end{aligned}$$

and use the calculus of residues to evaluate the explicit form of $G(x,y)$.

(b) For $c^2 \ge 2$ give an explicit example of a function $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ such that the equation

$$Lu = f \text{ in } \mathbb{R}$$

does not have a solution $u \in C^4(\mathbb{R}) \cap L^2(\mathbb{R})$.

Remark: (a) and (b) show that an operator $L^{-1}\colon C(\mathbb{R}) \cap L^2(\mathbb{R}) \to C^4(\mathbb{R}) \cap L^2(\mathbb{R})$ exists if and only if $0 \le c^2 < 2$

Problem 2 Find all *small-amplitude* traveling waves of

$$u_{tt} + u_{xxxx} = 1 - u^+ \text{ in } \mathbb{R} \times \mathbb{R},$$

i.e. all solutions of the form $u(x,t) = v(x - \omega t) + 1$, with $\|v\|_\infty \le 1$. Check that none of them is homoclinic, i.e. none of them satisfies $\lim_{x \to \pm\infty} v(x) = 0$.