

Aspects of nonlinear wave equations

Sheet 6

Problem 1

Here we show that the operator

$$\text{Id}: \begin{cases} H^1(\mathbb{R}) & \longrightarrow & L^q(K) \\ v & \longmapsto & v|_K \end{cases}$$

is compact for $1 \leq q \leq \infty$ and a compact subset $K \subset \mathbb{R}$.

Recall that an operator $A: X \rightarrow Y$ is called *compact* if for every bounded sequence $(v_k)_{k \in \mathbb{N}}$ in X the sequence $(Av_k)_{k \in \mathbb{N}}$ in Y is *precompact*, i.e. it has a convergent subsequence.

Use the following characterization due to Marcel Riesz: a bounded sequence $(v_k)_{k \in \mathbb{N}}$ in $L^q(K)$, $K \subset \mathbb{R}$ compact, $1 \leq q < \infty$, is precompact if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $h \in \mathbb{R}$, $|h| \leq \delta$ and all $k \in \mathbb{N}$

$$\int_K |v_k(x+h) - v_k(x)|^q dx \leq \varepsilon.$$

You can also use the fact that for $v \in H^1(\mathbb{R})$, $v'(x)$ exists for almost every $x \in \mathbb{R}$ in the classical sense and the fundamental theorem of calculus holds true for v .

Problem 2 Consider $v \in C_c^\infty(\mathbb{R})$, $v \neq 0$ and define the sequence $v_k(x) = k^\alpha v(\frac{x}{k})$, $k \in \mathbb{N}$.

- How do you choose $\alpha \in \mathbb{R}$ such that $\|v_k\|_{L^2(\mathbb{R})} = \|v\|_{L^2(\mathbb{R})}$?
- With this value of α verify that $(v_k)_{k \in \mathbb{N}}$ satisfies the conditions of Lemma 6 for all $q \in [2, \infty)$.
- Verify that $v_k \xrightarrow[k \rightarrow \infty]{} 0$ in $L^r(\mathbb{R})$ for $2 < r < \infty$ but not for $r = 2$.

Remark: This example shows that the condition $2 < r < \infty$ in Lemma 6 is sharp.

Notice the following swap:
08.06.2016, 14:00 - 15:30 - lecture
22.06.2016, 11:30 - 13:00 - exercise class