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Institute for Analysis
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Aspects of nonlinear wave equations

Sheet 7

Problem 1

In this exercise, follow Chapter 3 to obtain results for the wave-equation

$$(\star) \qquad u_{tt} - u_{xx} = f(u)$$

with $f(s) = -s + |s|^{p-1}\, s$, $p > 1$. Write $(\star)$ as a first order system

$$(\star\star) \qquad \begin{cases} u^1_t &=& u^2, \\ u^2_t &=& u^1_{xx} + f(u^1)\,. \end{cases}$$

(a) Formulate a variational problem to obtain homoclinic ground state profiles $v_\omega \in H^1(\mathbb{R})$ for traveling waves with speed $\omega \in (0, 1)$.

(b) Show that v_ω is unique up to shifting the argument and multiplication with -1 (*Hint:* phase plane and Theorem 3, Chapter 2).

(c) Formulate energy E and charge Q for $(\star\star)$.

(d) Consider

$$d(\omega) = \cancel{E\left[\begin{pmatrix} v_\omega \\ \omega v'_\omega \end{pmatrix}\right] - \omega Q\left[\begin{pmatrix} v_\omega \\ v'_\omega \end{pmatrix}\right]} = E\left[\begin{pmatrix} v_\omega \\ \omega v'_\omega \end{pmatrix}\right] - \omega Q\left[\begin{pmatrix} v_\omega \\ \omega v'_\omega \end{pmatrix}\right],$$

for ground states v_ω and show that $d(\omega) = C\sqrt{1-\omega^2}$ with a constant $C > 0$.

Note the difference to the fourth order problem: here $d(\omega)$ is strictly concave on $(-1, 1)$.

Problem 2

Recall the minimization problem for fixed $\omega \in [0, \sqrt{2})$: set $S_\lambda = \{\psi \in H^2(\mathbb{R}) : K[\psi] = \lambda\}$ for $\lambda > 0$ and

$$i_\lambda := \inf_{\psi \in S_\lambda} J[\psi] \quad \text{and} \quad \tilde{\mu}_\lambda := \frac{i_\lambda}{\lambda}$$

where $J[\psi] = \int_{\mathbb{R}} (\psi'')^2 - \omega^2(\psi')^2 + \psi^2\, dx$ and $K[\psi] = \int_{\mathbb{R}} |\psi|^{p+1}\, dx$. Consider also the ground state set

$$G_\lambda = \{v = \tilde{\mu}_\lambda^{\frac{1}{p-1}} \tilde{v}_\lambda : \tilde{v}_\lambda \text{ minimizes } J \text{ on } S_\lambda\}.$$

Show: $i_\lambda = \lambda^{\frac{2}{p+1}} i_1$ and $G_\lambda = G_1$. Hence the set of ground states does not depend on λ.

Notice the following swap: 08.06.2016, 14:00 - 15:30 - lecture 22.06.2016, 11:30 - 13:00 - exercise class