Aspects of nonlinear wave equations

Problem 1

In this exercise, follow Chapter 3 to obtain results for the wave-equation

\[(\star) \quad u_{tt} - u_{xx} = f(u)\]

with \(f(s) = -s + |s|^{p-1} s, \quad p > 1\). Write (\star) as a first order system

\[(\star\star) \quad \begin{cases} u_1^t = u_2, \\ u_2^t = u_1^{xx} + f(u_1). \end{cases}\]

(a) Formulate a variational problem to obtain homoclinic ground state profiles \(v_\omega \in H^1(\mathbb{R})\) for traveling waves with speed \(\omega \in (0, 1)\).

(b) Show that \(v_\omega\) is unique up to shifting the argument and multiplication with \(-1\) (\textit{Hint:} phase plane and Theorem 3, Chapter 2).

(c) Formulate energy \(E\) and charge \(Q\) for (\star\star).

(d) Consider

\[d(\omega) = E \left[ \frac{v_\omega}{\sqrt{1 - \omega^2}} \right] - \omega Q \left[ \frac{v_\omega}{\sqrt{1 - \omega^2}} \right],\]

for ground states \(v_\omega\) and show that \(d(\omega) = C \sqrt{1 - \omega^2}\) with a constant \(C > 0\).

Note the difference to the fourth order problem: here \(d(\omega)\) is strictly concave on \((-1, 1)\).

Problem 2

Recall the minimization problem for fixed \(\omega \in [0, \sqrt{2})\): set \(S_\lambda = \{ \psi \in H^2(\mathbb{R}) : K[\psi] = \lambda \}\) for \(\lambda > 0\) and

\[i_\lambda := \inf_{\psi \in S_\lambda} J[\psi] \quad \text{and} \quad \tilde{\mu}_\lambda := \frac{i_\lambda}{\lambda},\]

where \(J[\psi] = \int_\mathbb{R} (\psi')^2 - \omega^2 (\psi')^2 + \psi^2 \, dx\) and \(K[\psi] = \int_\mathbb{R} |\psi|^{p+1} \, dx\). Consider also the ground state set \(G_\lambda = \{ v = \tilde{\mu}_\lambda^{\frac{1}{p+1}} \tilde{v}_\lambda : \tilde{v}_\lambda \text{ minimizes } J \text{ on } S_\lambda \}\).

Show: \(i_\lambda = \lambda^{\frac{2}{p+1}} i_1\) and \(G_\lambda = G_1\). Hence the set of ground states does not depend on \(\lambda\).

Notice the following swap:
08.06.2016, 14:00 - 15:30 - lecture
22.06.2016, 11:30 - 13:00 - exercise class