Karlsruhe Institute of Technology (KIT)
Institute for Analysis
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Summer Semester 2016
15.6.2016

Aspects of nonlinear wave equations

Sheet 8

Problem 1

Here we consider complex-valued time-harmonic standing waves $u(x,t) = e^{\mathrm{i}\omega t}v(x)$ for

$$(\star) \qquad \begin{cases} u_{tt} - u_{xx} = -u + |u|^{p-1}\, u & \text{on } (a,b)\times\mathbb{R}, \\ u(a,t) = u(b,t) = 0 & \text{for } t \in \mathbb{R}, \end{cases}$$

where $p > 1$. Clearly u is $\frac{2\pi}{\omega}$-periodic in time. From the introduction to Chapter 4 we know that such standing waves exist for $|\omega| < 1$. Here we construct them for $|\omega| > 1$ in the following way:

(a) write the equation and the boundary condition for the profile v;

(b) make the ansatz $v(x) = (\omega^2-1)^\alpha q(\sqrt{\omega^2-1}\,x)$, and choose α in such a way that the equation for q does not contain ω;

(c) discuss the equation for q in the phase-plane and show that it has sign-changing periodic solutions for all periods belonging to the interval $(0, 2\pi)$;

(d) show that $(\star)$ has time-harmonic standing waves with profiles v which do not to change sign on (a,b) provided $(b-a)\sqrt{\omega^2-1} < \pi$;

(e) using (d) construct for all intervals (a,b) time-harmonic standing waves with (possibly sign-changing) profiles v;

(f) show that (d) still holds in the case $|\omega| = 1$.

Problem 2

Here we construct complex-valued standing waves $u(x,t) = e^{\mathrm{i}\omega t}v(x)$ for

$$(\star\star) \qquad \begin{cases} u_{tt} + u_{xxxx} = -u + |u|^{p-1}\, u & \text{on } (a,b)\times\mathbb{R} \\ u(a,t) = u(b,t) = u_x(a,t) = u_x(b,t) = 0 & \text{for } t \in \mathbb{R}, \end{cases}$$

where again $p > 1$.

The case $|\omega| < 1$:

(a) derive the equation and the boundary condition for the profile v;

(b) formulate a minimization problem from which you can derive the existence of a standing wave profile v, and show that a minimizer exists;
Hint: show that $\left(\int_a^b (\varphi'')^2 + (1-\omega^2)\varphi^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$ is an equivalent norm on $H_0^2(a,b)$.

The case $|\omega| \geq 1$:

(a) find conditions on ω for which $\left(\int_a^b (\varphi'')^2 + (1-\omega^2)\varphi^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$ is still an equivalent norm on $H_0^2(a,b)$.
Hint: consider the first eigenvalue λ_1 of $\varphi^{\mathrm{iv}} = \lambda\varphi$ on (a,b) with $\varphi(a) = \varphi'(a) = \varphi(b) = \varphi'(b) = 0$.

(b) for such ω proceed as in the previous case to show the existence of a standing wave profile;

(c) if you feel adventurous show that $\lambda_1 = \cancel{\left(\frac{\pi}{b-a}\right)^4} = \left(\frac{r_1}{b-a}\right)^4$, where $r_1 \in \mathbb{R}$ is the smallest, non-negative solution of the equation $\cos(r)\cosh(r) = 1$.

Recall:
22.06.2016, 11:30 - 13:00 - exercise class

Oral exams dates: 08.09.2016 (Thursday), 07.10.2016 (Friday)