Aspects of nonlinear wave equations

Problem 1

Here we consider complex-valued time-harmonic standing waves \( u(x,t) = e^{i\omega t}v(x) \) for

\[
\begin{aligned}
    u_{tt} - u_{xx} &= -u + |u|^{p-1}u \\
    u(a,t) &= u(b,t) = 0 
\end{aligned}
\]  

\((\star)\)

where \( p > 1 \). Clearly \( u \) is \( \frac{2\pi}{\omega} \)-periodic in time. From the introduction to Chapter 4 we know that such standing waves exist for \( |\omega| < 1 \). Here we construct them for \( |\omega| > 1 \) in the following way:

(a) write the equation and the boundary condition for the profile \( v \);

(b) make the ansatz \( v(x) = (\omega^2 - 1)^{\alpha}q(\sqrt{\omega^2 - 1}x) \), and choose \( \alpha \) in such a way that the equation for \( q \) does not contain \( \omega \);

(c) discuss the equation for \( q \) in the phase-plane and show that it has sign-changing periodic solutions for all periods belonging to the interval \((0, 2\pi)\);

(d) show that \((\star)\) has time-harmonic standing waves with profiles \( v \) which do not to change sign on \((a, b)\) provided \((b - a)\sqrt{\omega^2 - 1} < \pi\);

(e) using (d) construct for all intervals \((a, b)\) time-harmonic standing waves with (possibly sign-changing) profiles \( v \);

(f) show that (d) still holds in the case \( |\omega| = 1 \).

Problem 2

Here we construct complex-valued standing waves \( u(x,t) = e^{i\omega t}v(x) \) for

\[
\begin{aligned}
    u_{tt} + u_{xxxx} &= -u + |u|^{p-1}u \\
    u(a,t) &= u(b,t) = u_x(a,t) = u_x(b,t) = 0 
\end{aligned}
\]  

\((\star\star)\)

where again \( p > 1 \).

The case \( |\omega| < 1 \):

(a) derive the equation and the boundary condition for the profile \( v \);

(b) formulate a minimization problem from which you can derive the existence of a standing wave profile \( v \), and show that a minimizer exists;

\( \text{Hint:} \) show that \( \left( \int_a^b (\varphi')^2 + (1 - \omega^2)\varphi^2 \, dx \right)^{\frac{1}{2}} \) is an equivalent norm on \( H^2_0(a,b) \).
The case $|\omega| \geq 1$:

(a) find conditions on $\omega$ for which
\[
\left( \int_a^b (\varphi'')^2 + (1 - \omega^2)\varphi^2 \, dx \right)^{\frac{1}{2}}
\]
is still an equivalent norm on $H_0^1(a, b)$.

*Hint:* consider the first eigenvalue $\lambda_1$ of $\varphi^{iv} = \lambda \varphi$ on $(a, b)$ with $\varphi(a) = \varphi'(a) = \varphi(b) = \varphi'(b) = 0$.

(b) for such $\omega$ proceed as in the previous case to show the existence of a standing wave profile;

(c) if you feel adventurous show that $\lambda_1 = \left( \frac{r_1}{b-a} \right)^4 = \left( \frac{r_1}{b-a} \right)^4$, where $r_1 \in \mathbb{R}$ is the smallest, non-negative solution of the equation $\cos(r) \cosh(r) = 1$.

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**Recall:**
22.06.2016, 11:30 - 13:00 - exercise class

**Oral exams dates:** 08.09.2016 (Thursday), 07.10.2016 (Friday)