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Aspects of nonlinear wave equations

Sheet 9

Problem 1

In this exercise we consider, for $b \in \mathbb{R}$, the standing wave problem

$$\begin{cases} u_{tt} - u_{xx} + bu^+ &= f(x,t) \quad \big((x,t) \in \big(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\big) \times \mathbb{R}\big), \\ u\big(-\tfrac{\pi}{2}, t\big) = u\big(\tfrac{\pi}{2}, t\big) &= 0 \quad (t \in \mathbb{R}), \\ u(x, t+\pi) &= u(x,t) \quad \big((x,t) \in \big(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\big) \times \mathbb{R}\big), \\ u(-x,t) = u(x,t) &= u(x,-t) \quad \big((x,t) \in \big(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\big) \times \mathbb{R}\big). \end{cases} \qquad (\star)$$

Notice that we require u to be even in x and t.

(a) Consider first the linear problem

$$\begin{cases} v_{tt} - v_{xx} &= f(x,t) \quad \big((x,t) \in \big(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\big) \times \mathbb{R}\big), \\ v\big(-\tfrac{\pi}{2}, t\big) = v\big(\tfrac{\pi}{2}, t\big) &= 0 \quad (t \in \mathbb{R}), \\ v(x, t+\pi) &= v(x,t) \quad \big((x,t) \in \big(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\big) \times \mathbb{R}\big), \\ v(-x,t) = v(x,t) &= v(x,-t) \quad \big((x,t) \in \big(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\big) \times \mathbb{R}\big). \end{cases} \qquad (\star\star)$$

(i) Show $f \equiv 0$ implies $v \equiv 0$.

(ii) Compute all eigenvalues λ (i.e. replace $f(x,t)$ by $\lambda v(x,t)$).

(iii) Show that $\min\{|\lambda| : \lambda \text{ is an eigenvalue}\} = 1$.

(iv) For $f \in \mathcal{L}$ there exists a unique solution $v = Kf$ of $(\star\star)$, where

$$\mathcal{L} = \left\{\psi \in L^2_{\text{loc}}\left(\left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}\right) : \psi \text{ even in } x \text{ and } t, \pi\text{-periodic in } t\right\}.$$

Here $\left(\mathcal{L}, \|\cdot\|_{L^2(D)}\right)$ is a Hilbert space for $D = \big(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\big) \times (0, \pi)$.

(v) $K : \mathcal{L} \to \mathcal{L}$ is continuous with $\|K\| = 1$.

(b) With the help of K reformulate $(\star)$ as a fix-point problem

$$u = K\big(f - bu^+\big) \qquad (\star\star\star)$$

and show that for all $f \in \mathcal{L}$ there is unique solution of $(\star\star\star)$ if $|b| < 1$.

(c) For $|b| < 1$ compute the solution for $f(x,t) = \cos x$.