PROBLEM 1.1 26.04.2016 a

(a) DISPERSION RELATION $\Rightarrow$

$$\text{„}x_2<0\text{"} \quad -\frac{\omega^2}{c_-^2}+|k|^2=0 \quad \Leftarrow \quad k=\frac{\omega}{c_-}\begin{pmatrix}\sin\varphi\\ \cos\varphi\end{pmatrix}$$

$$-\frac{\omega^2}{c_-^2}+|k'|^2=0 \quad \Leftarrow \quad k'=\frac{\omega}{c_-}\begin{pmatrix}\sin\varphi'\\ -\cos\varphi'\end{pmatrix}$$

$$\text{„}x_2>0\text{"} \quad -\frac{\omega^2}{c_+^2}+|k''|^2=0 \quad \Leftarrow \quad k''=\frac{\omega}{c_+}\begin{pmatrix}\sin\varphi''\\ \cos\varphi''\end{pmatrix}$$

(b) CONTINUITY. $A e^{ik_1x_1}+A'e^{ik_1'x_1}=A''e^{ik_1''x_1} \quad (x_1\in\mathbb{R})$

α Lema 1 (26.04.2016 f) $\Rightarrow ik_1 = ik_1' = k_1''$

$$\frac{1}{c_-}\sin\phi=\frac{1}{c_-}\sin\phi'=\frac{1}{c_+}\sin\phi'' \quad \& \; A+A'=A''$$

$\phi,\phi'\in[0,\frac{\pi}{2})$ $\Downarrow$ $\phi=\phi'$

$$\Downarrow \quad \frac{c_+}{c_-}=\frac{\sin\phi''}{\sin\phi} \quad \text{SNELL'S LAW}$$

LAW OF REFRACTION

$$\frac{\partial v}{\partial x_1}=\begin{cases}Aik_1e^{ik\cdot x}+A'ik_1'e^{ik'\cdot x}, & x_2<0\\ A''ik_1''e^{ik''x}, & x_1>0\end{cases} \qquad \frac{\partial v}{\partial x_2}=\begin{cases}Aik_2e^{ik\cdot x}+A'ik_2''e^{ik'\cdot x}\\ A''ik_2''e^{ik''\cdot x}\end{cases}$$

$$\alpha \; \mathcal{C}^1 \Rightarrow \begin{cases}A+A'=A''\\ Ak_2+A'k_2'=A''k_2''\end{cases} \Leftrightarrow \begin{cases}-A'+A''=A\\ -\frac{A'k_2'}{k_2}+\frac{k_2''}{k_2}=A\end{cases}$$

$$\begin{pmatrix}-1 & 1\\ -\frac{k_2'}{k_2} & \frac{k_2''}{k_2}\end{pmatrix}\begin{pmatrix}A\\ A''\end{pmatrix}=A\begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}-1 & 1\\ 1 & \underbrace{\frac{-c_-\cos\phi''}{c_+\cos\phi}}_{=-\frac{tg\,\phi}{tg\,\phi''}}\end{pmatrix}\begin{pmatrix}A'\\ A''\end{pmatrix}=A\begin{pmatrix}1\\1\end{pmatrix}$$

26.04.2016 (b)

$$\begin{pmatrix} -1 & 1 \\ 1 & -\frac{tg\phi}{tg\phi''} \end{pmatrix}^{-1} = \frac{1}{\frac{tg\phi}{tg\phi''}-1} \begin{pmatrix} -\frac{tg\phi}{tg\phi''} & -1 \\ -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} A' \\ A' \end{pmatrix} = \frac{1}{\frac{tg\phi}{t\phi''}-1} \begin{pmatrix} \frac{-tg\phi}{tg\phi''} & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} A \\ A \end{pmatrix} = \frac{A}{tg\phi - tg\phi''} \begin{pmatrix} tg\phi + tg'' \\ 2tg\phi'' \end{pmatrix}.$$

c) $\psi \in C_c^\infty(\mathbb{R}^2)$; $R > 0$: $\operatorname{supp}\psi \subseteq B_R(0)$ $B_R^\pm(0) = B_R(0) \cap (\mathbb{R} \times \mathbb{R}^\pm)$

$$\int_{\mathbb{R}^2} \nabla v \cdot \nabla\psi \, dx = \int_{B_R(0)} \nabla \cdot v \cdot \nabla \cdot \psi dx = \int_{B_R^+(0)} \nabla v \cdot \nabla\psi \, dx + \int_{B_R^-(0)} \nabla v \cdot \nabla \psi \, dx =$$

$$= \oint_{\partial B_R^+(0)} \frac{\partial v}{\partial \nu} \cdot \psi d\sigma + \oint_{\partial B_R^-(0)} \frac{\partial v}{\partial \nu} \psi d\sigma - \int_{B_R^+(0) \cup B_R^-(0)} \Delta v \cdot \psi \, dx$$

GREEN'S FORMULA

$$\int_\Omega \nabla\alpha \cdot \nabla\beta \, dx = -\int_\Omega \alpha \cdot \Delta\beta \, dx + \int_{\partial\Omega} \frac{\partial\alpha}{\partial\nu} \cdot \beta \, d\sigma$$

$$= \int_{-R}^{R} -v_{x_2} \psi d\sigma + \int_{-R}^{R} v_{x_2} \psi d\sigma - \int_{B_R(0)} \Delta v \cdot \psi = \int_{B_R(0)} \frac{\omega^2}{c^2(x_2)} v \psi$$

PROBLEM 1.2 26.04.2016 C

(a)

(LWE) $\frac{1}{c^2(x_2)} u_{tt} - \Delta u = 0$ in $\mathbb{R}^2 \times \mathbb{R}$

INSERT $u(x,t) = e^{i(kx_1 - \omega t)} v(x_2)$ INTO (LWE):

$$\frac{1}{c^2(x_2)}(-1)\omega^2 e^{i(kx_1-\omega t)} v(x_2) - (-1)k^2 e^{i(kx_1-\omega t)} v(x_2) - e^{i(kx_1-\omega t)} v''(x_2) = 0$$

$$\left(-\frac{\omega^2}{c^2(x_2)} + k^2\right) v(x_2) - v''(x_2) \qquad (x_2 \in \mathbb{R}) \iff (*)$$

WEAK FORMULATION OF $(*)$:

$$\alpha v - v'' = 0$$

$$\psi \in C_c^\infty(\mathbb{R}) \quad \int_{\mathbb{R}} \alpha v \psi - v'' \psi = 0$$

$$\int_{\mathbb{R}} \alpha v \psi + v'\psi' = 0 \qquad (\psi \in C_c^\infty(\mathbb{R})) \quad (\because)$$

Let $v \in C^1(\mathbb{R})$, SOLVE POINTWISE $(*)$ IN $\mathbb{R}\setminus\{\pm a\}$; $\psi \in C_c^\infty(\mathbb{R})$

$R > a$: $[-R,R] \supseteq \operatorname{supp}\psi$.

$$\int_{\mathbb{R}} v'\psi' dx = \int_{-R}^{R} v'\psi' = \int_{[-R,R]\cap(-\infty,-a)} v'\psi' + \int_{(-R,R)\cap(-a,a)} v'\psi' + \int_{(a,\infty)\cap(-R,R)} v'\psi' =$$

$$= \underbrace{v'\psi\Big|_{-R}^{a} + v'\psi\Big|_{-a}^{a} + v'\psi\Big|_{a}^{R}}_{=0} - \int_{\substack{(-R,R)\cap(-\infty,a) \\ \cup (-R,R)\cap(-a,a) \\ \cup (a,\infty)\cap(-R,R)}} v''\psi\, dx = -\int_{\mathbb{R}} \alpha v \psi$$

$\Rightarrow (\because)$ HOLDS TRUE

(b) CONSIDER $(*)$ FOR $|x_2| > a$

$$v \text{ IS } \begin{cases} \text{OSCILATORY} & \text{IF } -\frac{\omega^2}{c_2^2} + k^2 < 0 \\ \text{EXPONENTIAL (decaying)} & \text{IF } -\frac{\omega^2}{c_2^2} + k^2 > 0 \end{cases}$$

WE WANT $v \xrightarrow{|x_2| \to \infty} 0 \Rightarrow -\frac{\omega^2}{c_2^2} + k^2 > 0$ (o)

MULTIPLY $(*)$ BY v AND INTEGRATE BY PARTS: $\int_{\mathbb{R}} \left(-\frac{\omega^2}{c^2} + k^2\right) v^2 + v'^2 dx = 0$

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(d)

cd (b) IF $-\frac{\omega^2}{c_1^2}+k^2>0 \Rightarrow V\equiv 0$.

$$-\frac{\omega^2}{c_1^2}+k^2\leqslant 0 \qquad (\bullet\bullet)$$

$$-\frac{\omega^2}{c_2^2}+k^2>0\geqslant -\frac{\omega^2}{c_1^2}+k^2 \qquad k^2\in\left(\frac{\omega^2}{c_1},\frac{\omega^2}{c_2}\right) \qquad (\bullet)\&(\bullet\bullet)$$

$$\frac{\omega^2}{c_1^2}>\frac{\omega^2}{c_2^2}$$

$$\boxed{c_2>c_1}$$

(c) IN $\mathbb{R}\setminus(-a,a)$ $\quad V=Be^{-\sqrt{k^2-\frac{\omega^2}{c_2^2}}|x|}$

IN $(-a,a)$ $\quad V=A\cos\left(\sqrt{\frac{\omega^2}{c_1^2}-k^2}\,x\right)$

(d)
$$\begin{cases} A\cos\sqrt{\frac{\omega^2}{c_1^2}-k^2}\,a=Be^{-\sqrt{k^2-\frac{\omega^2}{c_2^2}}\,a} \\ -A\cdot\sqrt{\frac{\omega^2}{c_1^2}-k^2}\cdot\sin\left(\sqrt{\frac{\omega^2}{c_1^2}-k^2}\,a\right)=-B\sqrt{k^2-\frac{\omega^2}{c_2^2}}\,e^{-\sqrt{k^2-\frac{\omega^2}{c_2^2}}\,a}\end{cases}$$

$$\begin{pmatrix}\cos\sqrt{\frac{\omega^2}{c_1^2}-k^2}\,a & -e^{-\sqrt{k^2-\frac{\omega^2}{c_2^2}}\,a} \\ \sqrt{\frac{\omega^2}{c_1^2}-k^2}\,\sin\left(\sqrt{\frac{\omega^2}{c_1^2}-k^2}\,a\right) & -\sqrt{k^2-\frac{\omega^2}{c_2^2}}\,e^{-\sqrt{k^2-\frac{\omega^2}{c_2^2}}\,a}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=0$$

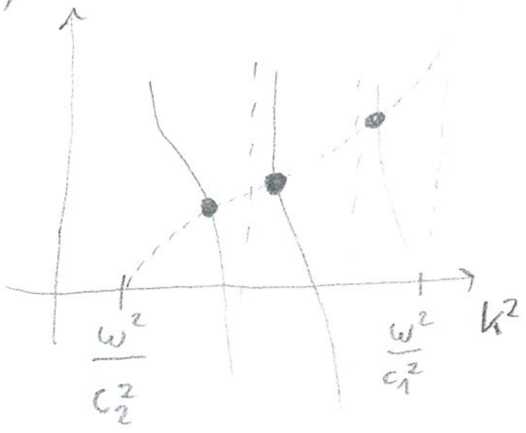
$\det(\ldots)=0 \iff$

$$\cos\sqrt{\frac{\omega^2}{c_1^2}-k^2}\,a\cdot\sqrt{k^2-\frac{\omega^2}{c_2^2}}\,e^{-\sqrt{k^2-\frac{\omega^2}{c_2^2}}\,a}=\sin\left(\sqrt{\frac{\omega^2}{c_1^2}-k^2}\,a\right)\cdot\sqrt{k^2-\frac{\omega^2}{c_2^2}}\cdot e^{-\sqrt{k^2-\frac{\omega^2}{c_2^2}}\,a}$$

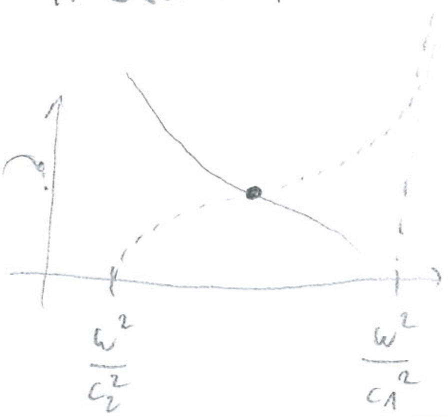
$$\mathrm{tg}\left(\sqrt{\frac{\omega^2}{c_1^2}-k^2}\,a\right)=\frac{\sqrt{k^2-\frac{\omega^2}{c_2^2}}}{\sqrt{\frac{\omega^2}{c_1^2}-k^2}} \qquad (**)$$

PROBLEM 1.2 26.04.2016 e

e)

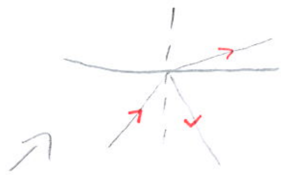


IF $0<a\ll 1$



TOTAL INTERNAL REFLECTION

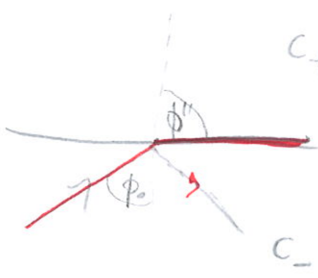
$$\frac{c_+}{c_-} = \frac{\sin\phi''}{\sin\phi} \qquad \sin\phi'' = \sin\phi \cdot \frac{c_+}{c_-}$$



ASSUME $\sin\phi'' = 1$ $\left(\phi'' = \frac{\pi}{2} = 90^\circ\right)$ & $c_- < c_+$ $\left\{\frac{c_-}{c_+} < 1\right\}$

$$\sin\phi = \frac{c_-}{c_+}$$

$$\phi_{CR} = \arcsin\frac{c_-}{c_+}$$ = CRITICAL ANGLE



IF $\phi > \phi_{CR} \Rightarrow \phi''$ „BECOMES COMPLEX"

EXPONENTIAL ATTENUATION

$$\sin\phi'' = \sin\phi \cdot \frac{c_+}{c_-} > 1$$

$$\cos\phi'' = \sqrt{1-\sin^2\phi''} = i\sqrt{\sin^2\phi - 1} \in i\mathbb{R}$$

$$A'' e^{i\frac{\omega}{c_+}\sin\phi'' x_1 - \frac{\omega}{c_+}\sqrt{\sin^2\phi - 1}\cdot x_2} e^{\omega t} \xrightarrow{x_2\to\infty} 0$$

EVANESCENT WAVE