P 11.1 12 07.2016 ⓐ

(a) $\psi^{IV} + m\psi = \lambda\psi$ on $(0, \pi)$, $\psi(0) = \psi''(0) = \psi(\pi) = \psi''(\pi) = 0$. $(*)$.

- NOTE THAT IF $\lambda_k = k^4 + m$ and $\psi_k(t) = \sin kt \Rightarrow (\lambda_k, \psi_k)$ SOLVES $(*)$.
- ANY $\psi : (0,\pi) \to \mathbb{R}$ SATISFYING $(*)$ can be WRITTEN AS

$$\psi(x) = \sum_{l=1}^{\infty} \gamma_l \sin lx.$$

IF $\psi$ SAT. $(*)$ $\Rightarrow$ $\gamma_l(l^4 + m) = \lambda \gamma_l$ $(l \in \mathbb{N})$

IF $\gamma_l \neq 0 \Rightarrow l^4 + m = \lambda$ $(l \in \mathbb{N})$.

$\Rightarrow$ ALL EIGENVALUES ARE OF THE FORM $\lambda_l = l^4 + m$

EIGEN FUNCTIONS $\psi_l(t) = \sin lt$

- FUNCTION $\mathbb{R} \ni s \mapsto \cos(\frac{2\pi}{T} s)$ IS $T$ PERIODIC $\{ \cos(\frac{2\pi}{T}(s+T)) = \cos(\frac{2\pi}{T}s + 2\pi) = \cos(\frac{2\pi}{T}s) \}$

  FREQUENCY $\omega = \frac{2\pi}{T}$

- FUNCTIONS $V_l(x,t) = \sin lx \cos(\sqrt{l^4 + m}\, t)$ ARE FREE VIBRATIONS FOR $L$.

$\omega_l = \sqrt{l^4 + m} = \sqrt{\lambda_l}$

$$Lv_l = (-(l^4 + m) + l^4 + m) \sin lx \cos\sqrt{l^4 + m}\, t = 0$$

$T_l = \frac{2\pi}{\omega_l}$

$V_l$ SATISFIES BOUNDARY CONDITIONS

- Let $V$ $[0,\pi) \times \mathbb{R} \to \mathbb{R}$ be SUCH THAT $LV = 0$ + B.C.

$$V(x,t) = \sum_{\substack{l \in \mathbb{N} \\ j \in \mathbb{N}_0}} \alpha_{lj} \sin lx \cos j\omega t \quad (x \in [0,\pi], t \in \mathbb{R}); \quad LV = \sum \alpha_{lj}(-j^2\omega^2 + l^4 + m) = 0$$

$$\Downarrow$$

$$j\omega = \sqrt{l^4 + m}$$

$$\omega_l = \sqrt{l^4 + m}$$

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(b) $m \in \mathbb{R} \setminus \mathbb{Q}$, $l_0 \in \mathbb{N}$ $\omega_{l_0} = \sqrt{l_0^4 + m}$. (cf lemma 4.2.10)

$$V_0(x,t) = \sum_{\substack{l \in \mathbb{N} \\ j \in \mathbb{N}_0}} \alpha_{jl} \sin l \cos jt$$

$$L_{\omega_{l_0}} V = 0 \Leftrightarrow \bigwedge_{l,j} \alpha_{jl} \neq 0 \Rightarrow l^4 + m - j^2 \omega_{l_0}^2 = 0$$

$$l^4 + m = j^2(l_0^2 + m)$$

$j = 1$ ↙ ⇘ $j \neq 1$

$l = l_0$ ok

$$m = \frac{l^4 - j^2 l_0^2}{j^2 - 1} \in \mathbb{Q} \ ↯$$

$$V(x,t) = \sin l_0 \cancel{x} \cdot \cos t = \psi_{l_0}(x) \cos t$$

P.11.1 (c). (cf. Lemma 4.2. 14). $\omega_1 = \sqrt{1+m}$ 12.07.2016 ⓑ

(A) THERE EXIST. UNCOUNTABLY many $m > 0$ SUCH THAT

$$S_\gamma^m = \{ \omega \in \mathbb{R} : |\omega j - \sqrt{m+l^2} \geqslant \frac{\gamma}{j} \quad (j \in \mathbb{N}, l \neq 1)\}$$

CONTAINS SEQUENCES $\omega_k^+, \omega_k^-$ $(k \in \mathbb{N})$ S.t.

$$\omega_k^- < \sqrt{m+1} < \omega_k^+ \quad \text{and} \quad \omega_k^\pm \xrightarrow[k\to\infty]{} \sqrt{m+1}$$

(B) THERE EXIST. UNCOUNTABLY MANY $m > 0$ SUCH THAT

$$\tilde{S}_\gamma^m = \{ \omega \in \mathbb{R} : |\omega j - \sqrt{m+l^4}| \geqslant \frac{\gamma}{j} \quad (j \in \mathbb{N}, l \neq 1)\}$$

CONTAINS SEQUENCES $\omega_k^+, \omega_k^-$ $(k \in \mathbb{N})$ S.t.

$$\omega_k^- < \sqrt{m+1} < \omega_k^+ \quad \text{and} \quad \omega_k^\pm \xrightarrow[k\to\infty]{} \sqrt{m+1}.$$

NOTE : (A) $\Longrightarrow$ (B). OBSERVE THAT $S_\gamma^m \subseteq \tilde{S}_\gamma^m$

$$\text{let } \mathcal{H} = \left\{ v \in H^{4,2}_{loc}(\Omega) : \begin{array}{l} v \text{ is } 2\pi\text{-PERIODIC \& EVEN IN TIME} \\ \text{trace } v|_{\{0\}\times\mathbb{R}} = 0 = \text{trace } v|_{\{\pi\}\times\mathbb{R}} \\ \text{trace } \frac{\partial^2 v}{\partial x^2}\Big|_{\{0\}\times\mathbb{R}} = 0 = \text{trace } \frac{\partial^2 v}{\partial x^2}\Big|_{\{\pi\}\times\mathbb{R}} \end{array} \right\}.$$

$$= \{ v = \sum_{\substack{l \in \mathbb{N} \\ j \in \mathbb{N}_0}} \alpha_{lj} \sin lx \cos jt : \sum (l^8 + j^4)\alpha_{lj}^2 < \infty \}.$$

Lemma ; (cf. ~~Lemma~~ PROP 4.2. 14

Let $\omega \in \tilde{S}_\gamma^m$ and $|\omega - \sqrt{1+m}| \leqslant \frac{\sqrt{1+m}}{4}$ FOR EVERY $f \in \mathcal{H} \cap K_1^\perp$ THERE EXISTS $v \in \mathcal{H} \cap K_1^\perp$ SUCH THAT $L_\omega v = f$ and $\|v\|_{H^{4,2}} \leqslant C \|f\|_{H^{4,2}}$.

i.e. $L_\omega^{-1} : \mathcal{H} \cap K_1^\perp \to \mathcal{H} \cap K_1^\perp$ IS a BOUNDED LINEAR OPERATOR.

PROOF.

$$v = \sum_{\substack{j \in \mathbb{N}_0, l \in \mathbb{N} \\ (j,l) \neq (1,1)}} \beta_{jl} \sin lx \cos jt, \quad f = \sum_{\substack{j \in \mathbb{N}_0, l \in \mathbb{N} \\ (j,l) \neq (1,1)}} \alpha_{jl} \sin lx \cos jt, \quad Lv = f \iff$$

$$\beta_{jl} = \frac{\alpha_{jl}}{-\omega^2 j + l^4 + m}; \quad \text{CONSIDER } \lambda_{jl} = -\omega^2 j^2 + l^4 + m = \quad (j, l \in \mathbb{N})$$

$$= (\sqrt{l^4+m} - \omega j)(\sqrt{l^4+m} + \omega j)$$

1° $l \neq 1$ NON-RESONANCE $\Longrightarrow$ $(\sqrt{l^4+m} - \omega j) > \frac{\gamma}{j}$

$$\omega = \omega - \omega_1 + \omega_1 \geqslant \omega_1 - \underbrace{|\omega - \omega_1|}_{\leq \frac{1}{4}\omega_1} \geqslant \frac{3}{4}\omega_1$$

$$|\lambda_{jl}| \geqslant \gamma\omega \geqslant \frac{3}{4}\gamma\omega_1 = \frac{3}{4}\gamma\sqrt{1+m}$$

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P.11.1.(c) $l=1 \Rightarrow j \geq 2$

$$|\sqrt{1+m} - \omega j| \geq |\sqrt{1+m} - \omega - (j-1)\omega| \geq \underbrace{(j-1)\omega}_{\geq \frac{3}{4}\omega_1} - \underbrace{|\overset{\omega_1}{\overset{\shortparallel}{\sqrt{1+m}}} - \omega|}_{\leq \frac{1}{4}\omega_1} \geq$$

$$\geq \frac{\omega_1}{4}(3(j-1)-1) \geq \frac{\omega_1}{2}$$

$$|\lambda_{i,j}| \geq \frac{\sqrt{1+m}}{2}$$

$\inf |\lambda_{\nu,\mu}| > 0$

THUS IF $f \in \mathcal{H} \Rightarrow v \in \mathcal{H}$.

and $\|v\|_{H^{4,2}} \leq C \|f\|_{H^{4,2}}$. $L_\omega v = f$.