P3.1.a FIRST INTEGRAL: $+u'''u' - \frac{u''^2}{2} + u'^2 + G(u)$ 03.05.2016

(d)

$G' = g$

IF u SOLVES $(**)$ $\Rightarrow$

$$\left(u'''u' - \frac{u''^2}{2} + u'^2 + G(u)\right)' = u^{IV}u' + \underline{u'''u''} - \underline{u''u'''} + 2u'u'' + g(u)u' =$$

$$= u'\left(\underbrace{u^{IV} + 2u'' + g(u)}_{\overset{(**)}{=}0}\right) = 0$$

P3.1.b

$$H(p_1,p_2,q_1,q_2) = p_2 q_1 - p_1 q_2 - \frac{p_2^2}{2} + G(q_1) - \frac{q_1^2}{2}$$

$$\left\{\begin{array}{ll} p_1 = -u' - u''' & q_1 = u \\ p_2 = u + u'' & q_2 = u' \end{array}\right\}$$

$$= (u+u'')u - (-u'-u''')u' - \frac{(u+u'')^2}{2} + G(u) - \frac{u'^2}{2}$$

$$= u^2 + u''u + u'^2 + u'''u' - \frac{u^2}{2} - uu'' - \frac{u''^2}{2} + G(u) - \frac{u'^2}{2}.$$

$$\begin{cases} \frac{d}{dx}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \frac{\partial H}{\partial q}(p,q) = \begin{pmatrix} p_2 + g(q_1) - q_1 \\ -p_1 \end{pmatrix} \\ \frac{d}{dx}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = -\frac{\partial H}{\partial p}(p,q) = -\begin{pmatrix} -q_2 \\ q_1 - p_2 \end{pmatrix} = \begin{pmatrix} q_2 \\ -q_1 + p_2 \end{pmatrix} \end{cases}$$

$\frac{\partial H}{\partial q_1}$ $\frac{\partial H}{\partial q_2}$ $\frac{\partial H}{\partial p_1}$ $\frac{\partial H}{\partial p_2}$

$\boxed{g(0)=0}$ $\Downarrow$ $((0,0),(0,0))$ IS AN EQUILIBRIUM POINT.

$$\frac{d}{dx}(-u'-u''') = u+u'' + g(u) - u \iff u^{IV} + 2u'' + g(u) = 0$$

$$\frac{d}{dx}(u+u'') = -(-u'-u''')$$

$$\frac{d}{dx}(u) = u'$$

$$\frac{d}{dx} u' = -u + u + u''$$

P.3.1.c

03.05.2016

b

THE MATRIX $M = \begin{pmatrix} \frac{\partial^2 H}{\partial p \partial q} & \frac{\partial^2 H}{\partial q^2} \\ -\frac{\partial^2 H}{\partial p^2} & -\frac{\partial^2 H}{\partial q \partial p} \end{pmatrix}$

$$M = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} g'(0)-1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \end{pmatrix}$$

$$\frac{\partial^2 H}{\partial p \partial q} = \begin{pmatrix} \frac{\partial^2 H}{\partial p_1 \partial q_1} & \frac{\partial^2 H}{\partial p_1 \partial q_2} \\ \frac{\partial^2 H}{\partial p_2 \partial q_1} & \frac{\partial^2 H}{\partial p_2 \partial q_2} \end{pmatrix} \qquad \frac{\partial^2 H}{\partial p^2} = \begin{pmatrix} \frac{\partial^2 H}{\partial p_1^2} & \frac{\partial^2 H}{\partial p_1 \partial p_2} \\ \frac{\partial^2 H}{\partial p_2 \partial p_1} & \frac{\partial^2 H}{\partial p_2^2} \end{pmatrix}$$

$$\frac{\partial^2 H}{\partial q^2} = \begin{pmatrix} \frac{\partial^2 H}{\partial q_1^2} & \frac{\partial^2 H}{\partial q_1 \partial q_2} \\ \frac{\partial^2 H}{\partial q_2 \partial q_1} & \frac{\partial^2 H}{\partial q_2^2} \end{pmatrix} \qquad \frac{\partial^2 H}{\partial q \partial p} = \begin{pmatrix} \frac{\partial^2 H}{\partial q_1 \partial p_1} & \frac{\partial^2 H}{\partial q_1 \partial p_2} \\ \frac{\partial^2 H}{\partial q_2 \partial p_1} & \frac{\partial^2 H}{\partial q_2 \partial p_2} \end{pmatrix}$$

$$\frac{\partial^2 H}{\partial p \partial q} = \left(\frac{\partial^2 H}{\partial q \partial p}\right)^T$$

$$\det(M - \lambda I) = \begin{vmatrix} -\lambda & -1 & g'(0)-1 & 0 \\ 1 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & -1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & g'(0)-1 & 0 \\ 0 & -\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2(\lambda^2+1) - 1\left(-(\lambda^2+1) + (g'(0)-1)(-1)\right) = (\lambda^2+1)^2 + g'(0) - 1$$

$$g'(0) = \frac{4}{\omega^4}$$

$$\lambda_{1/2} = \pm\sqrt{-1 + \sqrt{1-g'(0)}} = \pm\sqrt{-1+\sqrt{1-\frac{4}{\omega^4}}} = \pm i\sqrt{1-\sqrt{1-\frac{4}{\omega^4}}}$$

$$\lambda_{3/4} = \pm\sqrt{-1-\sqrt{1-g'(0)}} = \pm\sqrt{-1-\sqrt{1-\frac{4}{\omega^2}}} = \pm i\sqrt{1+\sqrt{1-\frac{4}{\omega^4}}}$$

IF $\omega > \sqrt{2} \Rightarrow \lambda_{1/2}, \lambda_{3/4} \in i\mathbb{R} \setminus \{0\}$.

$\lambda_{1/2} = \pm i\mu$ FOR SOME $\mu > 0$; $\lambda_{3/4} = \pm i\nu$, FOR SOME $\nu > 0$.

NOTE $\mu < \nu$.

$\Rightarrow \lambda_{1/2}$ IS NOT an INTEGER MULTIPLE OF $\lambda_{3/4}$.

THE LAPUNOV CENTER THM. APPLIES.

P3.2 04.05.2016

C

$$u_{tt} - u_{xx} = u(1-u^2) \xrightarrow[u(x,t)=V(x-\omega t),\ \omega\in(-1,1)\setminus\{0\}]{\text{TRAVELING WAVE}} (\omega^2-1)V'' = V(1-V^2) \qquad (\bullet)$$

$$V(s) = p\left(\frac{s}{\sqrt{1-\omega^2}}\right)$$

$$-p'' = p(1-p^2) \qquad (\bullet\bullet)$$

IF p SOLVES $(\bullet\bullet) \Rightarrow V$ SOLVES $(\bullet)$

FIRST INTEGRAL: $\frac{1}{2}p'^2 + \frac{p^2}{2} - \frac{p^4}{4}$

$\frac{1}{2}p'(s)^2 + F(p(s))$

$F(x) = \int x(1-x^2)\,dx = \frac{1}{2}x^2 - \frac{1}{4}x^4 + c$

VALUE OF FIRST INTEGRAL AT $\pm\infty$: $\frac{1}{4}$

LOOK FOR SOLUTION p: $\frac{1}{2}p'^2 = -\frac{p^2}{2} + \frac{p^4}{4} + \frac{1}{4}$

$$p'^2 = \frac{p^4}{2} - p^2 + \frac{1}{2} = \frac{1}{2}(p^4 - 2p^2 + 1) = \frac{1}{2}(p^2-1)^2$$

ASSUME: $p' > 0$

$$p' = \frac{1}{\sqrt{2}}(1-p^2)$$

$$\sqrt{2}\int_0^{p(x)} \frac{1}{1-p^2}\,dp = x$$

$\frac{1}{1-p^2} = \frac{1}{2}\left(\frac{1}{1-p} + \frac{1}{1+p}\right)$

$\int \frac{1}{1-s} = -\ln(1-s)$

$\int \frac{1}{1+s} = \ln(1+s)$

$\operatorname{arctgh} s = \frac{1}{2}\ln\frac{s+1}{1-s}$

$$\operatorname{arctgh} p(x) = \frac{x}{\sqrt{2}} \qquad p(x) = \operatorname{tgh}\frac{x}{\sqrt{2}}$$