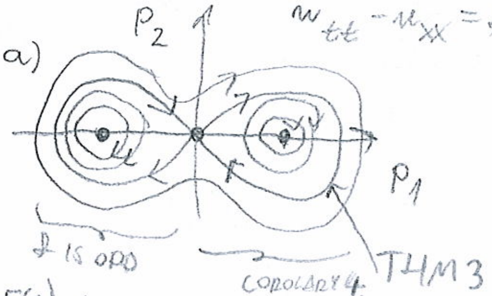
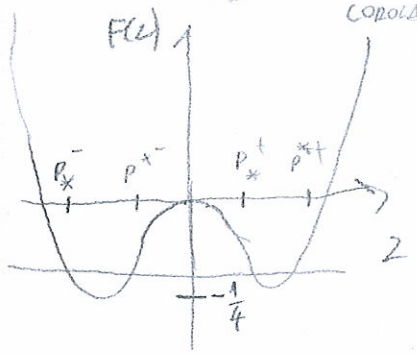
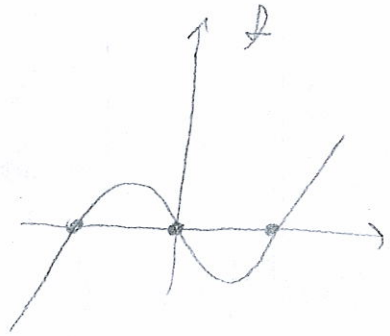
P 4.1 a) $u_{tt} - u_{xx} = f(u) \xrightarrow{u = V(x-wt)} (w^2-1)V'' = f(V)$ 13.05.2016

a

$\xrightarrow{w\in(-1,1)\setminus\{0\}} \quad -p'' = f(p) \Leftrightarrow V(s) = p\left(\frac{s}{\sqrt{1-w^2}}\right)$







$f(x) = (x+1)x(x-1) = x(x^2-1)$

$F(z) = \frac{z^4}{4} - \frac{z^2}{2} \geq -\frac{1}{4} \qquad F(\pm\sqrt{2}) = 0$

$f(x) = -f(-x)$
IF p SOLS $\Leftrightarrow$ THEN $\tilde{p} = -p$ ALSO

FIRST INTEGRAL: $H(p_1,p_2) = \frac{1}{2}p_2^2 + F(p_1) \geq -\frac{1}{4}$

FOR $c \in [-\frac{1}{4}, \infty)$ CONSIDER **LEVEL** SET

$G_c = \{(p_1,p_2) : H(p_1,p_2) = c\}.$

$c = -\frac{1}{4} \qquad G_c = \{(-1,0),(1,0)\}.$

$c \in (-\frac{1}{4}, 0) \qquad G_c = G_c^+ \cup G_c^-$

$G_c^- = \{(p_1,p_2) : p_*^- \leq p_1 \leq p^{*-}, p_2 = \pm\sqrt{c - F(p_1)}\}$

$G_c^+ = \{(p_1,p_2) : p_*^+ \leq p_1 \leq p^{*+}, p_2 = \pm\sqrt{c - F(p_1)}\}$

PERODIC SOLUTIONS.

$c = 0 \qquad G_0 = \{-\sqrt{2} \leq p_1 < 0 : p_2 = \pm\sqrt{-F(p_1)}\} \cup \{(0,0)\} \cup$

$\cup \{0 < p_1 \leq \sqrt{2} : p_2 = \pm\sqrt{-F(p_1)}\}$ HOMOCLINIC.

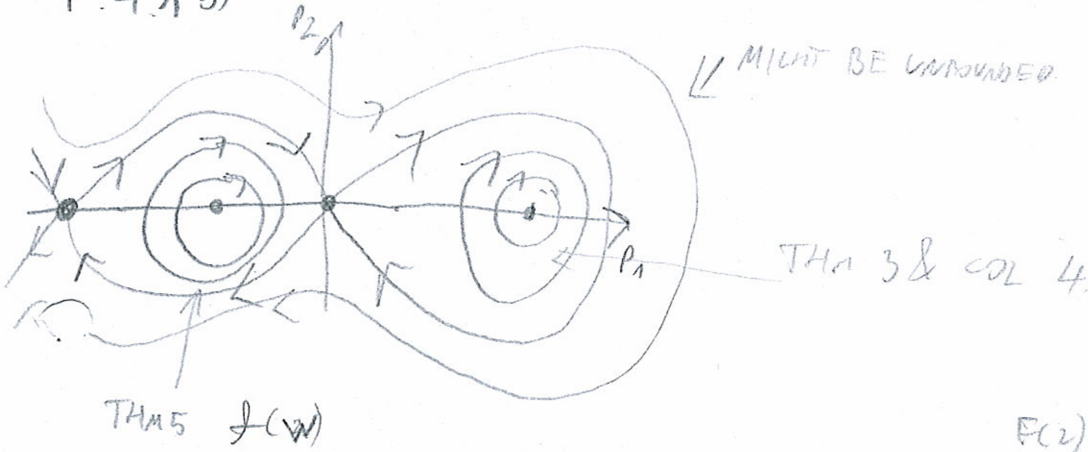
P. 4.1 a) 13.05.2016 (b)

$C > 0 \quad G_c = \{ (p_1, p_2) : \; p_* < p_1 < p^* \; : \; p_2 = \pm \sqrt{c - F(p_1)} \}.$

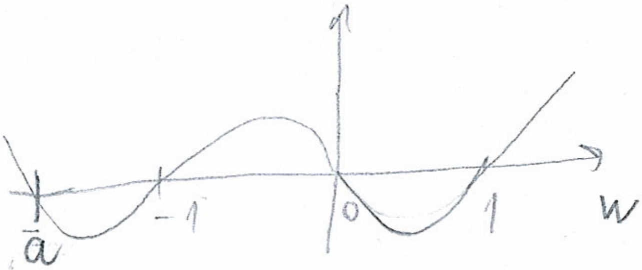
= PERIODIC ORBIT

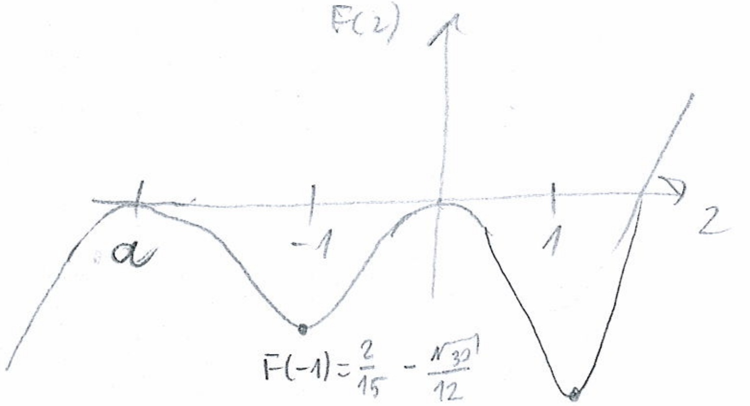
BOUNDED SET WITH NO EQUILIBRIUM.

P. 4.1 b)



$f(w)$





$F(-1) = \frac{2}{15} - \frac{\sqrt{30}}{12} \approx -0.3231$

$F(1) = -\frac{2}{15} - \frac{\sqrt{30}}{12} \approx -05898$

$f(w) = (w-a)(w+1)\,w\,(w-1)$

$= w^4 - w^2 - a w^3 + w a$

$F(w) = \frac{w^5}{5} - a\frac{w^4}{4} - \frac{w^3}{3} + \frac{w^2 a}{2} \qquad H(p_1, p_2) = \frac{1}{2} p_2^2 + F(p_1)$

HETEROCLINIC CONNECTION: a to 0 : $F(a) = F(0) = 0$

$a^5\left(\frac{1}{5} - \frac{1}{4}\right) + a^3\left(\frac{1}{2} - \frac{1}{3}\right) = 0$

$\boxed{a < -1}$ $a = 0$ OR $\frac{a^2}{20} = \frac{1}{6}$

$a = -\sqrt{\frac{10}{3}} < -1$

≈ -1.8258

P4.2 Burger's equation: ($a\in\mathbb{R}$ given constant) 11.05.2016

6

$$u_t + u u_x - a u_{xx} = 0 \quad \text{in } \mathbb{R}\times\mathbb{R} \qquad (*)$$

WE LOOK FOR TRAVELING WAVES, i.e. SOLUTIONS OF THE FORM:

$u(x,t) = v(x-\omega t)$

(*) BECOMES:

$$-\omega v' + v\cdot v' - a v'' = 0 \qquad (**)$$

INTEGRATE (**)

$$-\omega v + \frac{1}{2}v^2 - a v' = c_1$$

$$v' = \frac{1}{2a}\left(v^2 - 2\omega v - 2c_1\right)$$

$$\int \frac{v'}{v^2 - 2\omega v - 2c_1}\,dx = \int \frac{1}{2a}\,dx$$

$s = v(x)$

$ds = v'\,dx$

$$\int \frac{1}{s^2 - 2\omega s - 2c_1}\,ds = \frac{\underbrace{\operatorname{ar\,tgh}}_{= (tgh)^{-1}}\left(\frac{-s+\omega}{\sqrt{\omega^2+2c_1}}\right)}{\sqrt{\omega^2+2c_1}}$$

$$v(s) = -\sqrt{\omega^2+2c_1}\cdot tgh\left(\sqrt{\omega^2+2c_1}\left(\frac{1}{2a}s + c_2\right)\right) + \omega$$

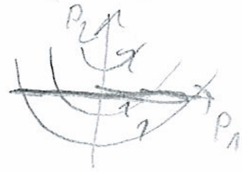
$$tgh(ix) = i\,tg\,x \xrightarrow{\omega^2+2c_1<0}$$ NO BOUNDED SOLUTIONS.

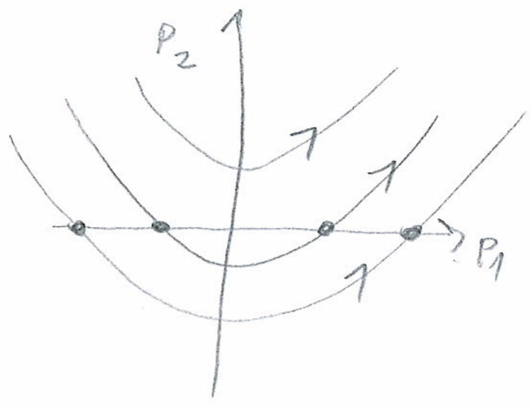
(**) as a 1st ORDER SYSTEM:

$p_1 = v \quad p_2 = v'$

$$\begin{cases} p_1' = p_2 \\ p_2' = \dfrac{p_2}{a}(p_1 - \omega) \end{cases} \qquad (***)$$

EQUILIBRUM: $p_2 = 0$





every point at p_1-axis is a stationary point

17.05.2016

d

$$\int \frac{dx}{ax^2+bx+c} = \dots$$

$$ax^2+bx+c = a(x-p)^2+q \quad p=\frac{-b}{2a} \quad q=\frac{-\Delta}{4a} \quad \Delta = b^2-4ac$$

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{(x-p)^2+\frac{q}{a}} \overset{s=x-p}{=} \frac{1}{a}\int \frac{ds}{s^2+\frac{q}{a}} \qquad \frac{q}{a} = \frac{-b^2+4ac}{4a^2}$$

$$\frac{1}{s^2+\tilde{q}} = \frac{1}{2\sqrt{-\tilde{q}}}\left(\frac{1}{s-\sqrt{-\tilde{q}}} - \frac{1}{s+\sqrt{-\tilde{q}}}\right) \quad \tilde{q} = \frac{q}{a}$$

$$\int \frac{ds}{s^2+\tilde{q}} = -\frac{1}{2\sqrt{-\tilde{q}}}\left(-\ln(s-\sqrt{-\tilde{q}}) + \ln(s+\sqrt{-\tilde{q}})\right) =$$

$$= -\frac{1}{\sqrt{-\tilde{q}}}\cdot\frac{1}{2}\ln\frac{s+\sqrt{-\tilde{q}}}{s-\sqrt{-\tilde{q}}} = -\frac{1}{\sqrt{-\tilde{q}}}\cdot\frac{1}{2}\ln\left(\frac{\frac{s}{\sqrt{-\tilde{q}}}+1}{\frac{s}{\sqrt{-\tilde{q}}}-1}\right)$$

$(\mathrm{ctgh})^{-1}$ $=$ $\mathrm{arctgh}\,x = \frac{1}{2}\ln\frac{x+1}{x-1}$

$$= -\frac{1}{\sqrt{-\tilde{q}}}\,\mathrm{arctgh}\frac{s}{\sqrt{-\tilde{q}}} = -\frac{1}{\sqrt{-\frac{q}{a}}}\,\mathrm{arctgh}\frac{s}{\sqrt{-\frac{q}{a}}}$$

$$\int \frac{dx}{ax^2+bx+c} = \dots = -\frac{1}{a\sqrt{-\frac{q}{a}}}\cdot\mathrm{arctgh}\left(\frac{x-p}{\sqrt{-\frac{q}{a}}}\right) =$$

$$= -\frac{2|a|}{a\sqrt{b^2-4ac}}\cdot\mathrm{arctgh}\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\cdot\frac{|a|}{a}\right) \quad \underline{\underline{\mathrm{arctgh}\,s = -\mathrm{arctgh}\,{-s}}}$$

$$= -\frac{2}{\sqrt{b^2-4ac}}\,\mathrm{arctgh}\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right) \quad \underline{\underline{\mathrm{arctgh}\,is = i\,\mathrm{arctg}\,s}}$$

$$= \frac{2}{\sqrt{4ac-b^2}}\,\mathrm{arctg}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)$$