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(a)

THM (RESIDUE THEOREM)

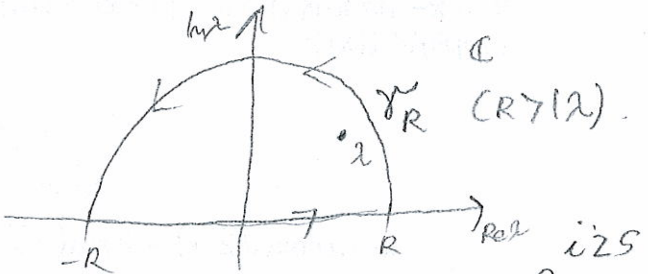
$U \subseteq \mathbb{C}$ SIMPLY CONNECTED DOMAIN, $a_1, \dots, a_n \in U$

$f : U \setminus \{a_1, \dots, a_n\} \to \mathbb{C}$ HOLOMORPHIC

γ is a CLOSED RECTIFIABLE CURVE in $U \setminus \{a_1, \dots, a_n\}$.

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{ind}_\gamma(a_k) \cdot \operatorname{Res}(f, a_k)$$

$s \in \mathbb{R}$ $\displaystyle\int_{\mathbb{R}} \frac{1}{k-\lambda} e^{iks}\,dk = ?$



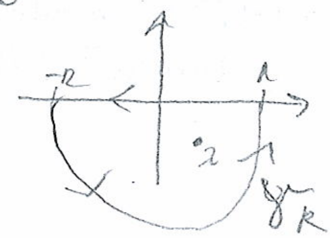
1° $\operatorname{Im}\lambda > 0$, $s > 0$

$\operatorname{Im} z > 0 \Rightarrow \operatorname{Re}(izs) < 0$ $\Rightarrow$ FOR $z \in \mathbb{C}$, $\operatorname{Im} z > 0$, $|z| = R$ $\displaystyle\frac{e^{izs}}{z-\lambda} \xrightarrow[z\to\infty]{} 0$

$$\int_{\mathbb{R}} \frac{1}{k-\lambda} e^{iks}\,dk = \lim_{R\to\infty} \int_{\gamma_R} \frac{1}{z-\lambda} e^{izs}\,dz$$

$$= 2\pi i \cdot \overset{\operatorname{ind}_{\gamma_R}(\lambda)}{1} \cdot \operatorname{Res}\left(\frac{1}{z-\lambda} e^{izs}\right) = 2\pi i \cdot e^{i\lambda s}$$

2° $\operatorname{Im}\lambda < 0$ $\displaystyle\int_{\mathbb{R}} \frac{1}{k-\lambda} e^{iks} = \lim_{R\to\infty} \frac{1}{z-\lambda} e^{izs}\,dz = 0$



3° $\operatorname{Im}\lambda < 0$ $s < 0$

$$\int_{\mathbb{R}} \frac{1}{k-\lambda} e^{iks}\,dk = \underset{\operatorname{ind}_{\gamma_R}(\lambda) = -1}{-}2\pi i\, e^{iks}$$

4° $\operatorname{Im}\lambda > 0$, $s < 0$ $\displaystyle\int_{\mathbb{R}} \frac{1}{k-\lambda} e^{iks}\,dk = 0$

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(5)

$$u^{IV} + \omega^2 u'' + u = f$$

TAKE FOURIER TRANSFORM

$$\widehat{u^{IV}} = k^4 \hat{u} \qquad \widehat{u''} = -k^2 \hat{u}$$ (cf LECTURE 18.05.2016 PROOF OF Lemma 2)

$$(k^4 - \omega^2 k^2 + 1)\hat{u} = \hat{f}$$

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\hat{f}(k)}{k^2 - \omega^2 k + 1} e^{ikx} dk$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f(\xi)}{k^4 - \omega^2 k^2 + 1} e^{ik(x-\xi)} dk\, d\xi$$

decompose: $\dfrac{1}{k^4 - \omega^2 k^2 + 1}$ $\qquad k^2 = h$

$$= \frac{1}{h^2 - \omega^2 h + 1} \qquad \mu^2 - \omega^2 \mu + 1 = 0$$

$$\mu_{1,2} = \frac{\omega^2}{2} \pm \sqrt{\frac{\omega^4}{4} - 1}$$

$$= \frac{\omega^2}{2} \pm \sqrt{1 - \frac{\omega^4}{4}}$$

$$\lambda = \mu_j \qquad \lambda = \alpha + \beta i$$

$$\lambda = \pm \frac{1}{2}\sqrt{2 + \omega^2} \pm i \frac{1}{2}\sqrt{2 - \omega^2}$$

	α	β		α	β
$\mu: +$	+	+	,	−	−
$\mu: -$	+	−	,	−	+

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$$k^2-\omega^2 k+1=(k-\mu)(k-\bar\mu) \qquad \mu=\frac{\omega^2}{2}+i\sqrt{1-\frac{\omega^4}{4}}$$

$$\frac{1}{k^4-\omega^2k^2+1}=\left(\frac{1}{k^2-\mu}-\frac{1}{k^2-\bar\mu}\right)\frac{1}{\mu-\bar\mu}$$

$$\frac{1}{k^2-\mu}=\frac{1}{k-\lambda_1}\cdot\frac{1}{k+\lambda_1} \qquad \lambda_1=\frac{1}{2}\sqrt{2+\omega^2}+\frac{i}{2}\sqrt{2-\omega^2}$$

$$\Rightarrow \frac{1}{k^2-\mu}=\frac{1}{2\lambda_1}\left(\frac{1}{k-\lambda_1}-\frac{1}{k+\lambda_1}\right)$$

$$\frac{1}{k^2-\bar\mu}=\frac{1}{k-\lambda_2}\cdot\frac{1}{k+\lambda_2} \qquad \lambda_2=\frac{1}{2}\sqrt{2+\omega^2}-\frac{i}{2}\sqrt{2-\omega^2}$$

$$\Rightarrow \frac{1}{k^2-\bar\mu}=\frac{1}{2\lambda_2}\left(\frac{1}{k-\lambda_2}-\frac{1}{k+\lambda_2}\right)$$

$$\frac{1}{k^4-\omega^2k^2+1}=\frac{1}{\mu-\bar\mu}\left[\frac{1}{2\lambda_1}\left(\frac{1}{k-\lambda_1}-\frac{1}{k+\lambda_1}\right)-\frac{1}{2\lambda_2}\left(\frac{1}{k-\lambda_2}-\frac{1}{k+\lambda_2}\right)\right]$$

IF $x>\xi$, $s>0$:

$$\int_{\mathbb{R}}\frac{e^{ik\overbrace{(x-\xi)}^{s}}}{k^4-\omega^2k^2+1}\,dk=\frac{1}{2\lambda_1}2\pi i e^{i\lambda_1(x-\xi)}+\frac{1}{2\lambda_2}2\pi i e^{-i\lambda_2(x-\xi)}$$

$$\overset{\lambda_1=\bar\lambda_2}{=}2i\,\mathrm{Re}\left(\frac{\pi}{\lambda_1}e^{i\lambda_1(x-\xi)}\right)$$

IF $x<\xi$, $s<0$:

$$\int_{\mathbb{R}}\frac{e^{ik(x-\xi)}}{k^4-\omega^2k^2+1}\,dk=\frac{1}{2\lambda_1}2\pi i e^{-i\lambda_1(x-\xi)}+\frac{1}{2\lambda_2}2\pi i e^{i\lambda_2(x-\xi)}$$

$$\overset{\bar\lambda_1=\lambda_2}{=}2i\,\mathrm{Re}\left(\frac{\pi}{\lambda_1}e^{-i\lambda_1(x-\xi)}\right)$$

$$\mathrm{Re}\left(\frac{1}{\lambda_1}e^{i\lambda_1 s}\right)=\mathrm{Re}\left(\frac{\bar\lambda_1}{|\lambda_1|^2}e^{i\lambda_1 s}\right)\doteq e^{-\sqrt{\frac{2-\omega^2}{2}}s}\,\mathrm{Re}\left(\bar\lambda_1 e^{i\sqrt{\frac{2+\omega^2}{2}}s}\right)$$

$$=e^{-\sqrt{\frac{2-\omega^2}{2}}s}\left(\frac{1}{2}\sqrt{2+\omega^2}\cos\sqrt{\frac{2+\omega^2}{2}}s+\frac{\sqrt{2-\omega^2}}{2}\sin\sqrt{\frac{2+\omega^2}{2}}s\right)$$

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$$u-\bar{u}=2i\sqrt{1-\frac{w^4}{4}}=i\sqrt{(2-w^2)(2+w^2)}$$

d)

$$u(x)=\int_{\mathbb{R}} f(\xi)\underbrace{\pi e^{-\frac{\sqrt{2-w^2}}{2}|x-s|}\left(\frac{\cos\frac{\sqrt{2+w^2}}{2}|x-\xi|}{2\sqrt{2-w^2}}+\frac{\sin\frac{\sqrt{2+w^2}}{2}|x-\xi|}{2\sqrt{2+w^2}}\right)}_{G(x,\xi)}d\xi$$

b)

Let $f=\begin{cases}1 & \text{on } [-1,1]\\ \text{something} & \\ 0 & \text{on } \mathbb{R}\setminus[-2,2]\end{cases}$

$u^{IV}+w^2u''+u=f$ IF $|x|>2$

u IS AS IN ex. P 5.2

$\Rightarrow \notin L^2$ ON $\mathbb{R}\setminus[-2,2]$.