P 6.1 31.05.2016 a

RECALL:

- $0<p<q\le\infty$, $|\Omega|<\infty \Rightarrow L^q(\Omega)\hookrightarrow L^p(\Omega)$ $\quad \|\cdot\|_{L^p}\le|\Omega|^{\frac1p-\frac1q}\cdot\|\cdot\|_{L^q}$ $\quad$ (*)

- COR 4 $\Rightarrow \bigwedge_{q\in[2,\infty]}\ \bigvee_{C_q>0}\ \bigwedge_{v\in H^1(\mathbb{R})} \|v\|_{L^q(\mathbb{R})}\le C_q\|v\|_{H^1(\mathbb{R})}$

- $|\Omega|<\infty \Rightarrow \bigwedge_{q\in[1,\infty]}\ \bigvee_{C_q>0}\ \bigwedge_{v\in H^1(\mathbb{R})} \|v\|_{L^q(\Omega)}\le\tilde{C}_q\|v\|_{H^1(\mathbb{R})}$

Let $(v_k)_{k\in\mathbb{N}}$ BE BOUNDED IN $H^1(\mathbb{R})$; i.e. $\|v_k\|_{H^1(\mathbb{R})}\le M$ $(k\in\mathbb{N})$

"$q\in[1,\infty)$"

$$\int_K |v_k(x+h)-v_k(x)|^q\,dx = \int_K \left|\int_x^{x+h} v_k'(s)\,ds\right|^q dx \le \int_K \left(|h|^{\frac12}M\right)^q dx = |K|\cdot|h|^{\frac q2}\cdot M^q$$

$I=[x,x+h]\cup[x+h,x]$, $\quad |I|=|h|$

$v_k'\in L^2(I) \overset{(*)}{\Rightarrow} v_k'\in L^1(I)$ $\quad \|v_k'\|_{L^1(I)}\le|h|^{1-\frac12}\|v_k'\|_{L^2(I)}\le|h|^{\frac12}\|v_k\|_{H^1(\mathbb{R})}$

$\le|h|^{\frac12}M$ $\quad$ (••)

"$q=\infty$"

RECALL (ARZELÀ-ASCOLI THM)

$I\subseteq\mathbb{R}$ BOUNDED INTERVAL, $f_n: I\to\mathbb{R}$ CONTINUOUS, $|f_n(x)|<M$ $(n\in\mathbb{N}, x\in I)$

$$\bigwedge_{\varepsilon>0}\ \bigvee_{\delta>0}\ \bigwedge_{\substack{x,y\in I\\|x-y|<\delta}}\ \bigwedge_{n\in\mathbb{N}} |f_n(x)-f_n(y)|<\varepsilon. \qquad (\bullet)$$

THEN $(f_n)_{n\in\mathbb{N}}$ HAS UNIFORMLY CONVERGENT SUBSEQUENCE.

WE NEED TO VERIFY (•)

$$|v_k(x)-v_k(y)| = \left|\int_y^x v'(s)\,ds\right| \le \int_y^x |v'(s)|\,ds \le M|x-y|^{\frac12}$$

P.6.2

30.05.2016

(b)

$$\|v_k\|^r_{L^r(\mathbb{R})} = k^{\alpha r}\int_{\mathbb{R}} v\left(\tfrac{x}{k}\right)^r dx = k^{\alpha r+1}\int_{\mathbb{R}} v\left(\tfrac{x}{k}\right)^r\cdot\frac{1}{k}\,dx$$

$$\xrightarrow[ds=\frac{1}{k}dx]{s=\frac{x}{k}} k^{\alpha r+1}\int_{\mathbb{R}} v(s)^r ds = k^{\alpha r+1}\cdot\|v\|_{L^r(\mathbb{R})}$$

$$\|v_k\|_{L^2(\mathbb{R})} = \|v\|_{L^2(\mathbb{R})} \overset{r=2}{\Longleftrightarrow} \alpha 2+1=0$$

$$\alpha = -\frac{1}{2}$$

$$r>2 \Rightarrow \quad -\frac{1}{2}r+1<0 \Rightarrow k^{-\frac{1}{2}r+1}\xrightarrow[k\to\infty]{} 0$$

$$\Rightarrow (v_k)_{k\in\mathbb{N}} \text{ IS BOUNDED IN } L^r(\mathbb{R}) \ (r\geq 2)$$

$$v_k \xrightarrow[k\to\infty]{L^r(\mathbb{R})} 0 \quad (r>2)$$

$$v_k'(x) = k^{-\frac{3}{2}} v'\left(\tfrac{x}{k}\right) \qquad \|v_k'\|_{L^r(\mathbb{R})} = k^{-3+1}\|v'\|_{L^r(\mathbb{R})} \xrightarrow[k\to\infty]{} 0$$

$$\Rightarrow (v_k) \text{ IS BOUNDED IN } H^1(\mathbb{R})$$

TAKE $R>0$:

$$S_k = \sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} |v_k|^q dx \leq k^{-\frac{1}{2}q}\cdot\sup_{y\in\mathbb{R}}|v(y)| \xrightarrow[k\to\infty]{} 0 \quad (q\in[2,\infty))$$