P 7.1 $\quad u_{tt} - u_{xx} = f(u) \qquad (\cdot)$ 06.06.2016

(a)

$u(x,t) = v(x - \omega t) \qquad f(s) = -s + |s|^{p-1}s \qquad F' = f$

$(\omega^2 - 1) v'' = f(v) \quad (:) \quad \}$ IF v SOLVES $(:) \Rightarrow u(x,t) = v(x - \omega t)$ SOLVES $(\cdot)$

$= -v + |v|^{p-1} \cdot v$

(a) $J_\omega[\psi] = \int_{\mathbb{R}} (1-\omega^2)(\psi')^2 + \psi^2 \, dx \qquad (\psi \in H^1(\mathbb{R}))$

$K[\psi] = \int_{\mathbb{R}} |\psi|^{p+1} dx \qquad (\psi \in H^1(\mathbb{R}))$

$S_\lambda = \{ \psi \in H^1(\mathbb{R}) : K[\psi] = \lambda \}$

$i_\lambda = \inf_{S_\lambda} J_\omega[\psi] \qquad \tilde{v}$ MINIMIZER

$J'_\omega[v]\varphi = 2\int_{\mathbb{R}} (1-\omega^2) v'\varphi' + v\varphi \, dx$

$K'_\omega[v]\varphi = (p+1)\int_{\mathbb{R}} |v|^{p-1} v \cdot \varphi \, dx$

LAGRANGE: $\lambda_1 J'_\omega(\tilde{v}) = \lambda_2 K'(\tilde{v})$

$$2\lambda_1 \int_{\mathbb{R}} (1-\omega^2)\tilde{v}'\varphi' + \tilde{v}\varphi \, dx = \lambda_2 (p+1) \int_{\mathbb{R}} |\tilde{v}|^{p-1} \tilde{v} \varphi \, dx$$

$\varphi = \tilde{v} \Rightarrow 2\lambda_1 J_\omega[\tilde{v}] = \lambda_2 (p+1) K[\tilde{v}] \longrightarrow 2\lambda_1 i_\lambda = \lambda_2 (p+1)\lambda$

$$2\lambda_1 \left(-(1-\omega^2)\tilde{v}'' + \tilde{v} \right) = \lambda_2 (p+1) |\tilde{v}|^{p-1} \tilde{v}$$

$$-(1-\omega^2)\tilde{v}'' + \tilde{v} = \tilde{\mu} |\tilde{v}|^{p-1} \tilde{v} \qquad \tilde{\mu} = \frac{i_\lambda}{\lambda} = \frac{\lambda_2 (p+1)}{2\lambda_1}$$

Let $v = \alpha \tilde{v} \qquad -(1-\omega^2) v'' + v = \underbrace{\tilde{\mu} \alpha^{1-p}}_{=1} |v|^{p-1} v$

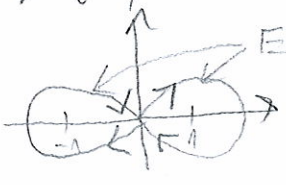
$$\alpha = \tilde{\mu}^{\frac{1}{p-1}}$$

$\Rightarrow v \in H^1$ IS a GROUND STATE.

(b) $v(x) = q\left(\frac{x}{\sqrt{1-\omega^2}} \right)$

$-q'' + q = |q|^{p-1} q \quad (\because) \}$ IF q SOLVES $(\because) \Rightarrow v(x) =$ SOLVES $(:)$

THM 3, CHAPTER 2



EXACTLY TWO HOMOCLINIC CURVES: ONE IS THE REFLECTION OF THE OTHER.

CURVE ON A PHASE PLANE $\Leftrightarrow$ SOLUTION OF ODE + ALL ITS TRANSLATIONS

P 7.1(c) 06.06.2016

(c) $E(u) = E\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \int_{\mathbb{R}} \frac{1}{2}(u^1_x)^2 + \frac{1}{2}(u^2)^2 - F(u^1)\, dx$ (b)

$$Q(\mathcal{U}) = Q\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \int_{\mathbb{R}} u^2 u^1_x \, dx$$

(**)

$$E'(u)\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \int_{\mathbb{R}} u^1_x \psi^1_x + u^2 \psi^2 - f(u^1)\psi^1 \, dx$$

$$Q'(\mathcal{U})\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \int_{\mathbb{R}} \psi^2 u^1_x + u^2 \psi^1_x \, dx$$

IF $\mathcal{U}$ SOLVES (**)

$$\frac{d}{dt} E(\mathcal{U}(t)) = E'(u(t)) \frac{d}{dt} \mathcal{U}(t) =$$

$$= \int_{\mathbb{R}} u^1_x \Big(\underset{\underset{u^2}{\parallel}}{\frac{du^1}{dt}}\Big)_x + u^2 \underbrace{\frac{du^2}{dt}}_{u^1_{xx} + f(u^1)} - f(u^1) \underbrace{\frac{du^1}{dt}}_{= u^2} \overset{\text{INTEGRATION}}{\underset{\text{BY PARTS}}{=}} 0$$

$$\frac{d}{dt} Q(\mathcal{U}(t)) = Q'(u(t)) \frac{d}{dt} \mathcal{U}(t) =$$

$$= \int_{\mathbb{R}} \underbrace{\frac{du^2}{dt}}_{u^1_{xx} + f(u^1)} \cdot u^1_x + u^2 \Big(\underbrace{\frac{du^1}{dt}}_{= u^2}\Big)_x =$$

$$= \int_{\mathbb{R}} u^1_{xx} u^1_x + f(u^1) u^1_x + u^2 u^2_x \, dx$$

$$= \int_{\mathbb{R}} \frac{1}{2} \frac{d}{dx} (u^1_x) + \frac{d}{dx} F(u^1_x) + \frac{1}{2} \frac{d}{dx} (u^2) \, dx = 0$$

P. 7.1 d) $G = \{ v_w = \tilde{w}^{\frac{1}{p-1}} \tilde{V}_w : \tilde{V}_w$ MINIMER OF J_w OVER $S_\lambda \}$ 14.06.2016

$$d(w) = E\begin{pmatrix} v_w \\ w v_w' \end{pmatrix} - w Q \begin{pmatrix} v_w \\ w v_w' \end{pmatrix} \qquad (v_w \in G)$$

as in lemma 14:

$$(?)\ d(w) = \left(\frac{1}{2} - \frac{1}{p+1}\right) J_w[v_w] = \left(\frac{1}{2} - \frac{1}{p+1}\right) K[v_w]$$

v_w IS a GROUND STATE:

$$(w^2-1) v_w'' \cancel{+} v_w = |v_w|^{p-1} v_w$$

$$\int_{\mathbb{R}} \underbrace{(w^2-1) v_w'' v_w}_{\shortparallel \atop (1-w^2)(v_w')^2} + v_w^2 = \int_{\mathbb{R}} |v_w|^{p+1}$$

$$J_w[w] \qquad = K[v_w] \qquad (\circ)$$

$$d(w) = E\begin{pmatrix} v_w \\ w v_w' \end{pmatrix} - w Q \begin{pmatrix} v_w \\ w v_w' \end{pmatrix} =$$

$$= \int_{\mathbb{R}} \frac{1}{2} v_w'^2 + \frac{1}{2} w^2 (v_w')^2 - F(v_w) - w^2 \int_{\mathbb{R}} v_w' \cdot v_w'$$

$$\frac{p+1-2}{2(p+1)} = \frac{p-1}{2(p+1)}$$

$$= \int_{\mathbb{R}} \frac{1}{2} (1-w^2)(v_w')^2 - F(v_w) = \frac{1}{2} J_w(v_w) - \frac{1}{p+1} K[v_w] \overset{(\circ)}{=} \overbrace{\left(\frac{1}{2} - \frac{1}{p+1}\right)} J_w(v_w)$$

$$= \left(\frac{1}{2} - \frac{1}{p+1}\right) K(v_w)$$

$v_w(x) = q\left(\frac{x}{\sqrt{1-w^2}}\right)$ (UP TO TRANSLATIONS & ± 1 FACTOR).

$$d(w) = \frac{p-1}{2(p+1)} K(v_w) = \frac{p-1}{2(p+1)} \int_{\mathbb{R}} \left| q\left(\frac{x}{\sqrt{1-w^2}}\right)\right|^{p+1} dx \overset{s = \frac{x}{\sqrt{1-w^2}}}{=}$$

$$= \sqrt{1-w^2} \cdot \underbrace{\frac{p-1}{2(p+1)} \int_{\mathbb{R}} |q(s)|^{p+1} ds}_{=: C}$$

p. 4.2 06.06.2016

(d)

NOTE $\psi \in S_\lambda \iff \lambda^{-\frac{1}{p+1}} \psi \in S_1$ $\quad \left\{ K[\psi] = \int_{\mathbb{R}} |\psi|^{p+1} dx \right.$

let $\tilde{v}_\lambda$ be a minimizer of J on S_λ

$$(\cdot)\ i_1 \le J[\lambda^{-\frac{1}{p+1}} \tilde{v}_\lambda] = \lambda^{-\frac{2}{p+1}} J[\tilde{v}_\lambda] = \lambda^{\frac{-2}{p+1}} i_\lambda$$

let $\tilde{v}_1$ be a minimizer of J on S_1.

$$(:)\ i_\lambda \le J[\lambda^{\frac{1}{p+1}} \tilde{v}_1] = \lambda^{\frac{2}{p+1}} J[\tilde{v}_1] = \lambda^{\frac{2}{p+1}} i_1$$

Hence

$$i_1 \overset{(\cdot)}{\le} \lambda^{-\frac{2}{p+1}} i_\lambda \overset{(:)}{\le} i_1$$

i.e. $\quad i_\lambda = \lambda^{\frac{2}{p+1}} i_1 \qquad (\because)$

Moreover

by $(\cdot)$ $\quad \lambda^{-\frac{1}{p+1}} \tilde{v}_\lambda$ is a minimizer of J on S_1 $\quad (\because)$

$(:)$ $\quad \lambda^{\frac{1}{p+1}} \tilde{v}_1$ — " — J on S_λ $\quad (::)$

with $m_\lambda = \{ \tilde{v}_\lambda$ minimizer of J on $S_\lambda \}$. $\quad \tilde{\mu}_\lambda = \frac{i_\lambda}{\lambda}$

$G_{\omega,\lambda} = \{ v = \tilde{\mu}_\omega^{\frac{1}{p-1}} \tilde{v}_\lambda :\ \tilde{v}_\lambda \in S_\lambda$ minimizer of $J_\omega \}$

we get $\quad \lambda^{-\frac{1}{p+1}} m_\lambda = m_1 \quad (\because)$

$$G_\lambda = \tilde{\mu}_\lambda^{\frac{1}{p-1}} \cdot m_\lambda = \frac{i_\lambda^{\frac{1}{p-1}}}{\lambda^{\frac{1}{p-1}}} \cdot m_\lambda \overset{(::)}{=} \frac{i_\lambda^{\frac{1}{p-1}}}{\lambda^{\frac{1}{p-1}}} \cdot \lambda^{\frac{1}{p+1}} \cdot m_1 \overset{(\because)}{=} \frac{i_1^{\frac{1}{p-1}} \cdot \lambda^{\frac{2}{p+1} \cdot \frac{1}{p-1}}}{\lambda^{\frac{1}{p-1}}} \lambda^{\frac{1}{p+1}} \cdot m_1 =$$

$$= \lambda^{\overbrace{\left(\frac{2}{p+1} \cdot \frac{1}{p-1} + \frac{1}{p+1} - \frac{1}{p-1}\right)}^{=0}} \cdot i_1^{\frac{1}{p-1}} \cdot m_1 = \tilde{\mu}_1^{\frac{1}{p-1}} \cdot m_1 = G_1$$