P8.1 20.06.2016 (d)

(a) $(*)\begin{cases} u_{tt} - u_{xx} = -u + |u|^{p-1}u & \text{on } (a,b)\times\mathbb{R} \\ u(a,t) = u(b,t) = 0 & (t\in\mathbb{R}) \end{cases}$

$$u(x,t) = e^{i\omega t} v(x)$$

$(\cdot)\begin{cases} -v'' + (1-\omega^2)v = |v|^{p-1}v \\ v(a) = v(b) = 0 \end{cases}$ IF V SOLVES $(\cdot)$ & u HAS ABOVE FORM $\Rightarrow$ u SOLVES $(*)$.

b) TRY $v(x) = (\omega^2-1)^{\alpha} q\left(\sqrt{\omega^2-1}\,x\right)$

$$-(\omega^2-1)^{\alpha+1}q'' + \underbrace{(1-\omega^2)(\omega^2-1)^{\alpha}}_{-(\omega^2-1)^{\alpha+1}} q = (\omega^2-1)^{\alpha p}|q|^{p-1}q$$

$$(\omega^2-1)^{\alpha+1} = (\omega^2-1)^{\alpha p}$$

$$\alpha+1 = \alpha p$$

$$\alpha = \frac{1}{p-1}$$

$$v(x) = (\omega^2-1)^{\frac{1}{p-1}} q\left(\sqrt{\omega^2-1}\,x\right)$$

$(:)\quad -q'' - q = |q|^{p-1}q \quad \& \quad q\left(\sqrt{\omega^2-1}\,a\right) = q\left(\sqrt{\omega^2-1}\,b\right) = 0$

IF q SOLVES $(:)$ & v HAS ABOVE FORM $\Rightarrow$ V SOLVES $(\cdot)$

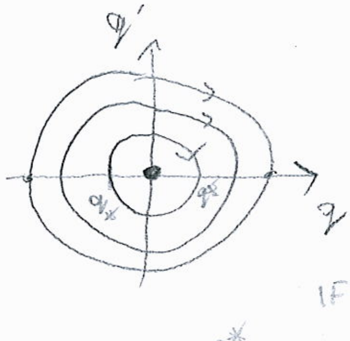
c) $\begin{cases} p_1' = p_2 \\ p_2' = \underbrace{-p_1 - |p_1|^{p-1}p_1}_{-f(p_1)} \end{cases}$ $\leftarrow (:)$ AS A FIRST ORDER SYSTEM

FIRST INTEGRAL $H(p_1,p_2) = \left(\frac{1}{2}p_2^2 + F(p_1)\right)\cdot 2$

$$H(p_1,p_2) = \left(\frac{1}{2}p_2^2 + \frac{1}{2}p_1^2 + \frac{1}{p+1}|p_1|^{p+1}\right)\cdot 2$$

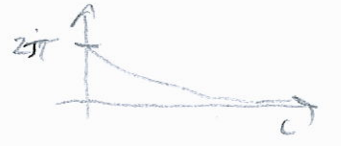
$$= p_2^2 + p_1^2 + \frac{2}{p+1}|p_1|^{p+1} = C > 0$$

$$= q'^2 + q^2 + \frac{2}{p+1}|q|^{p+1}$$



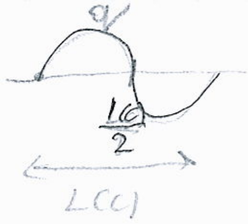
IF $q'(t) > 0$ $\quad q'(t) = \sqrt{C - q^2(t) - \frac{2}{p+1}|q(t)|^{p+1}}$

$$\int_{q_*}^{q^*} \frac{ds}{\sqrt{C - s^2 - \frac{2}{p+1}|s|^{p+1}}} \underset{ds = q'(t)dt}{\overset{s=q(t)}{=}} \int_0^{\frac{1}{2}L(C)} dt = \frac{1}{2}L(C)$$



$$L(C) \xrightarrow[C\to 0^+]{} \frac{2\pi}{\sqrt{f'(0)}} = 2\pi \quad (\text{cf. LECTURE } 02.05.2016)$$

S1(d)



IF $(b-a)\sqrt{\omega^2-1} = \frac{L(c)}{2}$ FOR SOME c

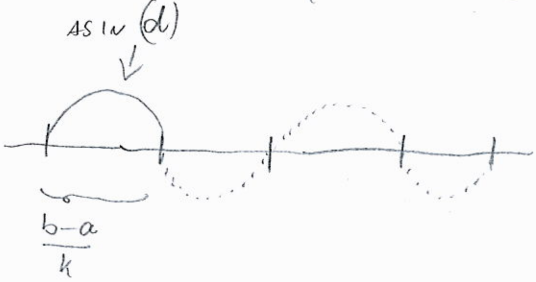
21.06.2016 (5)

then we have a PROFILE SAT. BOUNDARY COND.

HENCE IF $(b-a)\sqrt{\omega^2-1} < \pi \Rightarrow$ THERE EXISTS a TIME HARMONIC - STANDING WAVES WITH PROFILES V WHICH DO NOT CHANGE SIGN.

(e)

Let $k \in \mathbb{N}$: $\frac{b-a}{k}\sqrt{\omega^2-1} < \pi$



EXTEND IN AN ODD WAY.

(f) $\omega = 1$

$$-v'' = |v|^{p-1}v$$

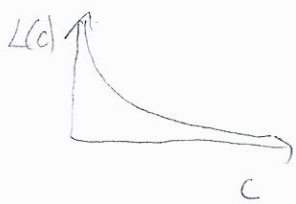
$$\begin{cases} p_1' = p_2 \\ p_2' = \underbrace{-|p_1|^{p-1}p_1}_{f(p_1)} \end{cases}$$

FIRST INTEGRAL $H(p_1,p_2) = p_2^2 + 2\cdot\frac{1}{p+1}|p_1|^{p+1} = c$

$$V'^2 + \frac{2}{p+1}|v|^{p+1} = c$$

IF $v' > 0$: $V' = \sqrt{c - \frac{2}{p+1}|v|^{p+1}}$

$L(c) \xrightarrow[c \to 0^+]{} \infty$



P8.2 $\{|\omega|<1\}$ 20.06.201

a)
$$(**)\begin{cases} u_{tt} + u_{xxxx} = -u + |u|^{p-1}u & \text{on } (a,b)\times\mathbb{R} \\ u(a,t) = u(b,t) = u_x(a,t) = u_x(b,t) = 0 & (t\in\mathbb{R}) \end{cases}$$

$$u(x,t) = e^{i\omega t}v(x)$$

$$(\cdot)\begin{cases} v^{iv} + (1-\omega^2)v = |v|^{p-1}v & \text{on } (a,b) \\ v(a) = v'(a) = v(b) = v'(b) = 0 \end{cases}$$

IF v SOLVES $(\cdot)$ & u HAS ABOVE FORM $\Rightarrow$ u SOLVES $(**)$.

(b) MINIMIZE $J(\varphi) = \int_a^b (\varphi'')^2 + (1-\omega^2)\varphi^2\,dx \quad (\varphi\in H_0^2(a,b))$

OVER $S = \{\varphi\in H_0^2(a,b) : K[\varphi] = \int_a^b |\varphi|^{p+1}dx = 1\}$.

$u\in H_0^1(a,b) \Rightarrow \|u\|_{L^2(a,b)} \le C\|u'\|_{L^2(a,b)}$

PROOF

W.L.O.S. $\varphi\in C_c^\infty(a,b)$ $\quad \varphi(x) = \int_a^x \varphi'(s)ds$

$$\|\varphi\|_{L^2}^2 = \int_a^b \varphi^2 dx = \int_a^b\left(\int_a^x \varphi'(s)ds\right)^2 \le \int_a^b\left(\int_a^x |\varphi'(s)|^2 ds\cdot\int_a^x 1\,dx\right) \le$$

$$\le \tilde{C}\int_a^b\int_a^b |\varphi'(s)|^2 dx \le C\|\varphi'\|_{L^2}^2$$

$\varphi_n \xrightarrow{H^2} u$

$\|u\|_{L^2(a,b)}^2 \le C\|u'\|_{L^2(a,b)}$

$\square$

$u\in H_0^2(a,b) \Rightarrow \|u\|_{L^2(a,b)} \le C\|u'\|_{L^2(a,b)}$

$u'\in H_0^1(a,b) \Rightarrow \|u'\|_{L^2(a,b)} \le C\|u''\|_{L^2(a,b)}$ $\Rightarrow \|u\|_{L^2} + \|u'\|_{L^2} \le C\|u''\|$

$\|u\|_{H^2(a,b)} \le C\|u''\|_{L^2}$

$\varphi\in H_0^2(a,b)$

$$\|\varphi\|_{H_0^2(a,b)}^2 \le C\|\varphi''\|_{L^2(a,b)}^2 \le C J(\varphi) \le C\|\varphi\|_{H_0^2(a,b)}^2$$

$\Rightarrow$ J IS WEAK-LOWER SEMICONTINUOUS. $\quad$ $v_k \rightharpoonup v \Rightarrow \|v\| \le \liminf_{k\to\infty}\|v_k\|$

$J(v) \le \liminf_{k\to\infty}\|v_k\|$

$i = \inf_{\varphi\in S} J[\varphi]$

$v_k\in S : J[v_k] \to i$

v_k IS BOUNDED $\Rightarrow$ v_k HAS WEAKLY CONVERGENT SUBSEQUENCE; W.L.O.S. $v_k \rightharpoonup \tilde{v}$.

$H_0^1(a,b) \overset{\text{compact}}{\hookrightarrow} L^q(a,b) \Rightarrow$ v_k HAS STRONGLY CONVERGENT SUBSEQUENCE IN L^{p+1}, W.L.O.S. v_k CONVERGES IN L^q TO $\tilde{v}$.

$\Rightarrow$ MINIMIZER EXISTS.

P8.2 20.06.2016

LAGRANGE MULTIPLIER

$$J'[\varphi]\psi = 2\int_a^b \varphi''\psi'' + (1-\omega^2)\varphi\psi$$

ⓓ

$$\alpha_1 J'(\tilde{v}) = \alpha_2 K'(\tilde{v}) \qquad K'[\varphi](\psi) = (p+1)\int_a^b |\varphi|^{p-1}\varphi\psi$$

$$2\alpha_1 \int_a^b \tilde{v}''\psi'' + (1-\omega^2)v\psi = \alpha_2 (p+1)\int_a^b |\tilde{v}|^{p-1}\tilde{v}\psi$$

$$\psi = \tilde{v} \Rightarrow 2\alpha_1 \underbrace{J[\tilde{v}]}_{=i} = {}_2(p+1)\underbrace{K[\tilde{v}]}_{=1} \Rightarrow 2\alpha_1 i = \alpha_2 (p+1)$$

$$2\alpha_1\left(\tilde{v}^{iv} + (1-\omega^2)\tilde{v}\right) = \alpha_2 (p+1)|\tilde{v}|^{p-1}\tilde{v} \qquad \tilde{\mu} = \frac{\alpha_2(p+1)}{2\alpha_1} = i$$

$$\tilde{v}^{iv} + (1-\omega^2)\tilde{v} = \tilde{\mu}\,|\tilde{v}|^{p-1}\tilde{v}$$

$$v = \alpha\tilde{v}$$

$$v^{iv} + (1-\omega^2)\tilde{v} = \underbrace{\tilde{\mu}\cdot\alpha^{1-p}}_{=1}|v|^{p-1}v$$

$$\alpha = \tilde{\mu}^{\frac{1}{p-1}}$$

P8.2

$\{|\omega| \geq 1\}$

(a) $$\lambda_1 = \min_{\varphi\in H_0^2(a,b)} \frac{\int_a^b \varphi''^2\,dx}{\int_a^b \varphi^2\,dx} > 0$$

$$\lambda_1 \int_a^b \varphi^2\,dx \leq \int_a^b \varphi''^2\,dx$$

$$J[\varphi] = \int_a^b \varphi''^2 + (1-\omega^2)\varphi^2\,dx \geq \int_a^b \varepsilon\varphi''^2 + \underbrace{(\lambda_1(1-\varepsilon) + 1-\omega^2)}_{>0}\varphi^2\,dx$$

$$\Leftrightarrow \omega^2 < 1 + \lambda_1$$

(b) PROCEED AS BEFORE.