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ⓐ

FUNCTION $s \mapsto \cos 2ls$ IS π PERODIC $(l \in \mathbb{Z})$

FUNCTION $s \longmapsto \cos(2m+1)s$: $\cos(2m+1)\cdot\frac{\pi}{2} = \cos m\pi + \frac{\pi}{2} = 0$

$\cos(2m+1)-\frac{\pi}{2} = 0$ $(m \in \mathbb{Z})$.

a)

$$V(x,t) = \sum_{l,m \in \mathbb{N}_0} V_{l,m} \cdot \cos(2m+1)x \cdot \cos(2l\cdot t)$$

$$V \in \mathcal{L} \Leftrightarrow \underbrace{\sum_{l,m \in \mathbb{N}_0} |V_{l,m}|^2}_{\overset{\parallel}{\|V\|_{L^2}}} < \infty$$

$\leftarrow$ PARSEVAL'S IDENTITY

(ii) $$V_{tt} - V_{xx} = \sum_{l,m \in \mathbb{N}_0} V_{l,m}\left(-4l^2 + (2m+1)^2\right)\cos(2m+1)x \cdot \cos(2l\cdot t) \quad (\bullet)$$

$$V_{tt} - V_{xx} = \lambda V \Rightarrow -4l^2 + (2m+1)^2 = \lambda =: \lambda_{l,m}$$

eigen function : $\cos(2m+1)x \cdot \cos(2l\cdot t)$.

(iii) NOTE $\lambda_{l,m} \in \mathbb{Z}$

SUPPOSE $\lambda_{l,m} = 0$ FOR SOME $l,m \in \mathbb{N}_0 \Rightarrow \underbrace{(2m+1)^2}_{\overset{\cap}{2\mathbb{N}+1}} = \underset{\overset{\cap}{2\mathbb{N}}}{4l^2}$ $\lightning$

(iii) $\lambda_{0,0} = 1$

THEREFORE $\min\{|\lambda| : \lambda \text{ e.value}\} = 1$.

(i) SINCE $\lambda_{l,m} \neq 0$ $(l,m \in \mathbb{N}_0) \Rightarrow f \equiv 0 \Rightarrow V \equiv 0$.

(i) $\Rightarrow$ UNIQUENESS

(iv) WRITE $f(x,t) = \sum_{l,m \in \mathbb{N}_0} f_{l,m} \cos(2m+1)x \cdot \cos(2l\cdot t)$.

$$V_{tt} - V_{xx} = f \overset{(\bullet)}{\Longrightarrow} V_{l,m}\left(-4t^2 + (2m+1)^2\right) = f_{l,m} \quad (l,m \in \mathbb{N}_0)$$

$$V_{l,m} = \frac{f_{l,m}}{(-4t^2 + (2m+1)^2} \quad (l,m \in \mathbb{N}_0)$$

$\sum \frac{1}{\lambda_{l,m}^2} < \infty$

$f \in \mathcal{L} \Rightarrow V \in \mathcal{L}$

(v) $\|Kf\| = \|V\| = \sum_{l,m \in \mathbb{N}} \frac{\|f_{l,m}\|^2}{\underbrace{|\lambda_{l,m}|^2}_{\geq 1}} \leq \|f\|_{L^2} \Rightarrow \|K\| \leq 1$

LET f be an eigen FUNCTION $\sim \lambda_{0,0}$ $\Big\} \Rightarrow \|K\| = 1$

$Kf = \frac{1}{\lambda_{00}} f \Rightarrow \|Kf\| = \|f\|$

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(b) IF $u \in \mathcal{L}$: $u = K(f - bu^+)$ $\Rightarrow$ u SOLVES $(**)$. ⓑ

$$\|K(f - bu^+) - K(f - bv^+)\| = \|K(b(-u^+ + v^+))\| \leq b\|u^+ - v^+\| \leq \underset{<1}{b}\|u - v\|$$

$\Rightarrow$ BANACH'S FIXED POINT THM.

u IS UNIQUE. (FOR ALL $f \in \mathcal{L}$)

(c) Let $u(x,t) = \alpha \cos x \geq 0$ $(x \in (-\frac{\pi}{2}, \frac{\pi}{2}))$

$$u_{tt} - u_{xx} + bu^+ \quad + \alpha \cos x + b\alpha \cos x = \alpha(b+1)\cos x = \cos x = f(x,t)$$

$$\alpha(b+1) = 1$$

$$\alpha = \frac{1}{b+1} > 0$$

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(:) Ⓒ

$$|\lambda_{l,m}| = |(2m+1)^2 - 4l^2| \geq |m| + |l|$$

$$\|$$

$$\underbrace{|2m+1-2l|}_{\geq 1}\,\underbrace{|2m+1+2l|}_{\geq 1}.$$

$$V^K = \sum_{0 \leq l,m \leq K} \frac{f_{l,m}}{\lambda_{l,m}} \cos(2m+1)x \cdot \cos 2lt \qquad (K \in \mathbb{N} \cup \{\infty\})$$

$$f^K = \dots$$

$$V^K \xrightarrow[K\to\infty]{L^2} V, \quad f^K \xrightarrow[K\to\infty]{L^2} f \quad (∷) \quad \text{NOTE: } V^K_{tt} - V^K_{xx} = f^K \quad (K \in \mathbb{N}) \quad (:\cdot)$$

$$\frac{\partial V}{\partial x} = \sum_{l,m} -\frac{f_{l,m}}{\lambda_{l,m}} \cdot (2m+1) \sin(2m+1) \cdot \cos 2lt$$

$$\frac{\partial V}{\partial t} = \sum_{l,m} -\frac{f_{l,m}}{\lambda_{l,m}} 2l \cos(2m+1) \sin 2lt.$$

$$(:) \Rightarrow \sum_{l,m} \left|\frac{f_{l,m}}{\lambda_{l,m}} 2l\right|^2 < \infty \;\&\; \sum_{l,m} \left|\frac{f_{l,m}}{\lambda_{l,m}} (2m+1)\right|^2 < \infty.$$

$V \in W^{1,2}(D)$,, WEAK SOLUTION!!

$$R^{a,b} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times [a,b] \qquad S = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}$$

$$W^{1,2}_{*,0}(R^{a,b}) = cl_{\|\cdot\|_{W^{1,2}(R)}} \{\varphi \in C_c(R^{a,b})\}$$

$$W^{1,2}_{*,0,PER}(D) = \{\varphi \in W^{1,2}_{*,0}(R^{a,b}) \; (a,b>0) \;\&\; \varphi \; \pi\text{-PERIODIC IN } t\}$$

$$= cl_{\|\cdot\|_{W^{1,2}(D)}} \{\varphi \in C^{\infty}_c(S) : \varphi : \pi\text{-PERIODIC IN } t\}.$$

$$(:\cdot) \Rightarrow \int_D -V^K_t \varphi_t + V^K_x \varphi_x = \int_D f^K \varphi$$

$$K \to \infty. \quad (∷) \Rightarrow \int_D -V_t \varphi_t + V_x \varphi_x = \int_D f \varphi.$$

V IS A WEAK SOLUTION OF (**