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Aspects of Nonlinear Wave Equations
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– Summary (1st Version of July 18th, 2016) –

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1. Linear wave equations

— not part of this summary —

2. Traveling waves for $u_{tt} - u_{xx} = f(u)$

We consider traveling waves for the following nonlinear wave equation

$$u_{tt} - \Delta u = f(u) \text{ in } \mathbb{R}^n \times \mathbb{R}. \tag{3}$$

A solution of the form $u(x,t) = v(k \cdot x - \omega t)$ with a non-constant profile $v : \mathbb{R} \to \mathbb{R}$ is called a *traveling wave*. The profile v has to satisfy the following equation:

$$(\omega^2 - |k|^2)v'' = f(v) \text{ in } \mathbb{R} \tag{4}$$

Remark. Without loss of generality we may assume $|k| = 1$. Moreover, if u solves (3) and $A \in O(n)$ then $u(Ax, t)$ also solves (3). Hence there is no preferred direction for traveling waves and again without loss of generality we may assume $k = (1, 0, \dots, 0)$.

Theorem 3 (homoclinic profile). *Let $f \in C^1(\mathbb{R})$ satisfy*

(i) $f(0) = 0$, $f'(0) < 0$,
(ii) *there is $z_1 > 0$ such that $F(z_1) > 0$, where $F(z) := \int_0^z f(w)\,dw$ for $z \in \mathbb{R}$*
(iii) *if $z_0 > 0$ is the first positive zero of F then $f(z_0) \neq 0$.*

Then there exists a unique solution $p \in C^2(\mathbb{R})$ of

$$-p'' = f(p) \tag{5}$$

with the properties

(a) $p > 0$, $p(-s) = p(s)$, $p(0) = z_0$
(b) $0 < p(s) \leq C_1 e^{-C_2|s|}$ *for all $s \in \mathbb{R}$ and some $C_1, C_2 > 0$.*

For all $\omega \in (-1,1) \setminus \{0\}$ and for all $k \in \mathbb{R}^n$, $|k| = 1$ there exist traveling waves of (3) of the form

$$u(x,t) = v(k \cdot x - \omega t)$$

where $v(s) = p(\frac{s}{\sqrt{1-\omega^2}})$.

Corollary 4 (periodic profile)**.** *In addition to the assumptions* (i)–(iii) *of Theorem 3 suppose*

(iv) *f has exactly one zero $z^* \in (0, z_0)$ and $f'(z^*) > 0$.*

Then (3) *has infinitely many periodic traveling waves. More precisely, if $\omega \in (-1,1) \setminus \{0\}$ and $k \in \mathbb{R}^n$, $|k| = 1$ are given, then for all $\tau \in \left(\frac{2\pi\sqrt{1-\omega^2}}{\sqrt{f'(z^*)}}, \infty\right)$ there exists a traveling wave*

$$u(x,t) = v(k \cdot x - \omega t)$$

such that v is τ-periodic, i.e., $u(x + \tau k, t) = u(x,t)$, $u(x, t + \frac{\tau}{\omega}) = u(x,t)$.

Lyapunov's center theorem *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a C^3-function. We write $H = H(p,q)$ with $p, q \in \mathbb{R}^n$. Consider the Hamiltonian system*

$$\left\{ \begin{array}{rcl} \dot{p} &=& \frac{\partial H}{\partial q}(p,q), \\ \dot{q} &=& -\frac{\partial H}{\partial p}(p,q). \end{array} \right. \tag{6}$$

Suppose the matrix

$$\begin{pmatrix} \dfrac{\partial^2 H}{\partial p \partial q} & \dfrac{\partial^2 H}{\partial q^2} \\ -\dfrac{\partial^2 H}{\partial p^2} & -\dfrac{\partial^2 H}{\partial q \partial p} \end{pmatrix}$$

at an equilibrium point $(p_0, q_0) \in \mathbb{R}^{2n}$ has a pair of simple eigenvalues $\pm i\lambda$, $\lambda > 0$ and no other integer multiple of $\pm i\lambda$ is an eigenvalue. Then in a neighbourhood of (p_0, q_0) the equation (6) *has periodic solutions. More precisely, for $\epsilon \in (0, \epsilon_0)$ there are periodic solutions $(p_\epsilon, q_\epsilon) : \mathbb{R} \to \mathbb{R}^{2n}$ with period $L(\epsilon)$ such that $(p_\epsilon, q_\epsilon) \to (p_0, q_0)$ and $L(\epsilon) \to \frac{2\pi}{\lambda}$ as $\epsilon \to 0$.*

For the traveling-wave purpose of Corollary 4

$$p_1' = p_2, \qquad p_2' = -f(p_1)$$

take $H(p_1, p_2) = \frac{1}{2}p_2^2 + F(p_1)$. At the equilibrium z^* the matrix

$$\begin{pmatrix} \dfrac{\partial^2 H}{\partial p_1 \partial p_2} & \dfrac{\partial^2 H}{\partial p_2^2} \\ -\dfrac{\partial^2 H}{\partial p_1^2} & -\dfrac{\partial^2 H}{\partial p_2 \partial p_1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -f'(z^*) & 0 \end{pmatrix}$$

has the eigenvalue $\pm i\lambda$ with $\lambda = \sqrt{f'(z^*)}$.

Theorem 5 (heteroclinic profile). *Let $f \in C^1(\mathbb{R})$ satisfy:*

(i) $f(0) = 0,\ f'(0) > 0$,
(ii) *there exist* $a, b \in \mathbb{R}$ *with* $a < 0 < b$ *with* $f(a) = f(b) = 0,\ f'(a) < 0,\ f'(b) < 0$,
(iii) *except* 0 *there is no other zero of* f *in* (a, b),
(iv) $F(a) = F(b)$ *where* $F(z) = \int_0^z f(w)\, dw$.

Then there exists a unique solution $p \in C^2(\mathbb{R})$ *of* (5) *with the properties*

(a) $p(0) = 0,\ p'(s) > 0$ *for all* $s \in \mathbb{R}$
(b) $0 < p(s) - a \le C_1 e^{C_2 s},\ 0 < b - p(s) \le C_1 e^{-C_2 s}$

For all $\omega \in (-1, 1) \setminus \{0\}$ *and for all* $k \in \mathbb{R}^n$, $|k| = 1$ *there exist traveling waves of* (3) *of the form*

$$u(x, t) = v(k \cdot x - \omega t)$$

where $v(s) = p(\frac{s}{\sqrt{1-\omega^2}})$.

3. Traveling waves for $u_{tt} + u_{xxxx} = f(u)$

Let $p > 1$. We consider traveling waves for the fourth-order nonlinear wave equation

$$u_{tt} + u_{xxxx} = -u + |u|^{p-1}u \text{ in } \mathbb{R}^2. \tag{1}$$

The traveling wave ansatz $u(x, t) = v(x - \omega t)$ leads to the following equation for the profile v:

$$v^{(iv)} + \omega^2 v'' = -v + |v|^{p-1}v \text{ in } \mathbb{R} \tag{2}$$

3.1. Existence of traveling waves of (1).

Theorem 1. *For every* $\omega \in (-\sqrt{2}, \sqrt{2}) \setminus \{0\}$ *there exists at least one traveling wave* $u(x, t) = v(x - \omega t)$ *with a homoclinic profile function* v *such that* $v(x) \to 0$ *as* $x \to \pm\infty$.

The proof of existence works as follows: define

$$J_\omega[\psi] = \int_{\mathbb{R}} (\psi'')^2 - \omega^2(\psi')^2 + \psi^2\, dx, \quad \psi \in H^2(\mathbb{R}),$$

$$K[\psi] = \int_{\mathbb{R}} |\psi|^{p+1}\, dx, \quad \psi \in H^2(\mathbb{R})$$

and

$$S_1 = \{\psi \in H^2(\mathbb{R}) : K[\psi] = 1\}.$$

Proposition 8. *Let* $0 \le |\omega| < \sqrt{2}$ *and* $p > 1$. *Then on the set* S_1 *the functional* J_ω *has a minimizer, i.e. there exists* $\tilde{v} \in S_1$ *such that* $J_\omega[\tilde{v}] \le J_\omega[\psi]$ *for all* $\psi \in S_1$.

Remarks.
1) Let $\tilde{\mu} = J_\omega[\tilde{v}] = \min_{S_1} J_\omega$ and set $v = \tilde{\mu}^{\frac{1}{p-1}}\tilde{v}$. Then v satisfies (2). It is called a ground state of (2). The set

$$G_\omega = \{v = \tilde{\mu}^{\frac{1}{p-1}}\tilde{v} : \tilde{v} \text{ is a minimizer of } J_\omega|_{S_1}\}$$

is called the set of ground states of (2).
2) We could have obtained the ground states also by minimizing J_ω over the set $S_\lambda = \{\psi \in H^2(\mathbb{R}) : K[\psi] = \lambda\}$, $\lambda > 0$. In this case, we rescale the minimizer $\tilde{v}$ by $\tilde{\mu} = \frac{J_\omega[\tilde{v}]}{\lambda} = \frac{\min_{S_\lambda} J_\omega}{\lambda}$.

3.2. **Stability of traveling waves of** (1)**.**

Rewrite (1) as a system. For this let $f(s) = -s + |s|^{p-1}s$ and $F(s) = -\frac{s^2}{2} + \frac{|s|^{p+1}}{p+1}$.

$$\begin{cases} u^1_t &= u^2 \\ u^2_t &= -u^1_{xxxx} + f(u^1) \end{cases}$$

Definition 11. *A continuous function* $U = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} : [0,T) \to X := H^2(\mathbb{R}) \times L^2(\mathbb{R})$ *is called a weak solution provided*

$$\frac{d}{dt}\int_{\mathbb{R}} u^1\psi^1 + u^2\psi^2\,dx = \int_{\mathbb{R}} u^2\psi^1 - u^1_{xx}\psi^2_{xx} + f(u^1)\psi^2\,dx \text{ for all } \Psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \in L^2(\mathbb{R}) \times H^2(\mathbb{R}).$$

Definition 12 (E=energy, Q=charge)**.**

$$E : \begin{cases} H^2(\mathbb{R}) \times L^2(\mathbb{R}) & \to \mathbb{R} \\ U = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} & \mapsto \int_{\mathbb{R}} \frac{1}{2}(u^1_{xx})^2 + \frac{1}{2}(u^2)^2 - F(u^1)\,dx \end{cases}$$

$$Q : \begin{cases} H^2(\mathbb{R}) \times L^2(\mathbb{R}) & \to \mathbb{R} \\ U = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} & \mapsto \int_{\mathbb{R}} u^2 u^1_x\,dx \end{cases}$$

Assumption. *For every* $\kappa > 0$ *there exists a time* $T = T(\kappa)$ *such that: if* $\left\| \begin{pmatrix} u^1_0 \\ u^2_0 \end{pmatrix} \right\|_X \le \kappa$ *then there exists a weak solution* $U : [0,T) \to X$ *with*

$$U(0) = \begin{pmatrix} u^1_0 \\ u^2_0 \end{pmatrix}, \qquad E(U(t)) = E(U(0)), \qquad Q(U(t)) = Q(U(0)).$$

Definition 13. *Fix* $\omega \in [0, \sqrt{2})$ *and let* G_ω *be the set of ground states. For* $v \in G_\omega$ *let* $V = (v, \omega v')$ *and define*

$$d(\omega) = E[V] - \omega Q[V].$$

Remark. The fact that d is a function of ω and does not depend on which element v from the set of ground states we take, is a result that one has to prove (it follows e.g. from Lemma 13).

Theorem 24. *Suppose the function d is strictly convex in an interval around ω. Then the set G_ω is stable, i.e., for every $\epsilon > 0$ there exists $\delta > 0$ with the following property: if* $\operatorname{dist}(U_0, G_\omega) \le \delta$ *then the weak solution U with $U(0) = U_0$ exists on $[0, \infty)$ and satisfies*

$$\sup_{0<t<\infty} \operatorname{dist}(U(t), G_\omega) \le \epsilon.$$

Remark. In some cases one knows that G_ω only consists of a single function v and all its shifts $v(\cdot + \tau)$, $\tau \in \mathbb{R}$. In this case the *stability* of G_ω as described in the above theorem is also called *orbital stability* of v.

4. Standing, time periodic waves

4.1. Variational approach to standing waves.

$$\begin{cases} u_{tt} - u_{xx} &= -|u|^{p-1}u \quad \text{on } (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) &= 0 \quad \text{for all } t \in \mathbb{R}, \\ u(x, t+T) &= u(x,t) \quad \text{for all } (x,t) \in (0,\pi) \times \mathbb{R}. \end{cases} \tag{2}$$

Theorem 1. *Let $p > 1$. If $\frac{T}{\pi} \in \mathbb{Q}$ then there exists a non-trivial real-valued T-periodic generalized solution of* (2).

Remarks.
(a) The solution is time-dependent, since (2) does not have any time-independent, non-trivial solutions.
(b) Definition 7 below explains what a generalized solution is.

The proof of Theorem 1 (given only in the case $T = 2\pi$) relies on an explicit representation formula (Lemma 3 below) for solutions of the linear problem:

$$\begin{cases} u_{tt} - u_{xx} &= f(x,t) \quad \text{on } (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) &= 0 \quad \text{for all } t \in \mathbb{R}, \\ u(x, t+2\pi) &= u(x,t) \quad \text{for all } (x,t) \in (0,\pi) \times \mathbb{R}. \end{cases} \tag{3}$$

For $f = 0$ the null-space of all generalized solutions is given by

$$N = \{q(t+x) - q(t-x) : q \text{ is } 2\pi\text{-periodic}, q \in L^1(0, 2\pi), \int_0^{2\pi} q\, dx = 0\}.$$

Lemma 3. *Let $\Omega = (0,\pi)\times(0,2\pi)$ and let $f : (0,\pi)\times\mathbb{R} \to \mathbb{R}$ be 2π-periodic in t, $f \in L^1(\Omega)$ and $f \perp_{L^2} N$. Then the unique generalized solution u of (3) with $u \perp_{L^2} N$ is given as follows*

$$u(x,t) = \psi(x,t) + q(t+x) - q(t-x)$$

where

$$\psi(x,t) = \frac{-1}{2}\int_x^\pi \int_{t-(y-x)}^{t+(y-x)} f(y,s)\,ds\,dy + c\frac{\pi - x}{\pi}$$

$$c = \frac{1}{2}\int_0^\pi \int_{t-y}^{t+y} f(y,s)\,ds\,dy$$

$$q(t) = \frac{-1}{2\pi}\int_0^\pi \psi(x,t-x) - \psi(x,t+x)\,dx.$$

Theorem 4. *For f as in Lemma 3 let $Kf := u$. Then the operator K has the following properties*

(a) $K : N^\perp \to C(\overline{\Omega})$ *is bounded,*
(b) $K : N^\perp \cap L^q(\Omega) \to C^\alpha(\overline{\Omega})$ *is bounded with* $\alpha = 1 - \frac{1}{q}$, $1 < q \leq \infty$,
(c) $K : N^\perp \cap C^{k,\alpha}(\overline{\Omega}) \to C^{k+1,\alpha}(\overline{\Omega})$ *is bounded,* $k = 0, 1$,
(d) $K : N^\perp \cap L^2(\Omega) \to N^\perp \cap L^2(\Omega)$ *is a compact, self-adjoint operator with eigenvalues* $\{\frac{1}{j^2-k^2} : j,k \in \mathbb{N}_0, j \neq k\}$,
(e) $K : N^\perp \cap L^q(\Omega) \to C(\overline{\Omega})$ *is compact for* $1 < q \leq \infty$.

A necessary condition for having a solution u of (2) with $T = 2\pi$ is $-|u|^{p-1}u \in N^\perp$. However, this is a nonlinear condition on u. One way to overcome this is to define $v := -|u|^{p-1}u$. This relation can be inverted because the odd p-th power is strictly monotone. I.e., $u = -|v|^{r-1}v$ with $r = 1/p$. Now (2) with $T = 2\pi$ can be rewritten as

$$-|v|^{r-1}v = u = Kv, \qquad v \in L^{r+1}(\Omega) \cap N^\perp. \tag{4}$$

Notice that the condition $-|u|^{p-1}u \in N^\perp$ has now been transferred into v belonging to the linear space $L^{r+1}(\Omega) \cap N^\perp$.

The idea how to solve (4) comes form the calculus of variations (cf. Section 3.1). Define the Fréchet-differentiable functionals

$$J[v] = \int_\Omega vKv\,d(x,t), \quad v \in L^{r+1}(\Omega) \cap N^\perp,$$

$$M[v] = \int_\Omega |v|^{r+1}\,d(x,t), \quad v \in L^{r+1}(\Omega) \cap N^\perp$$

and

$$S = \{v \in L^{r+1}(\Omega) \cap N^\perp : M[v] = 1\}.$$

Lemma 6. *On the set S the functional J has a minimizer, i.e. there exists $\tilde{v} \in S$ such that $J[\tilde{v}] \leq J[v]$ for all $v \in S$.*

Remark. Let $\mu = J[\tilde{v}] = -\min_S J$. Then $\mu < 0$ and if we set $v := (-\mu)^{\frac{1}{p-1}}\tilde{v}$ then v satisfies

$$(**) \qquad \int_\Omega (Kv)\phi\, d(x,t) = -\int_\Omega |v|^{r-1}v\phi\, d(x,t) \text{ for all } \phi \in L^{r+1}(\Omega) \cap N^\perp.$$

If v satisfies $(**)$ then it is not quite a solution of (4) – but almost. The final step in the proof of Theorem 1 is the following: set $u := -|v|^{r-1}v$ and define $\psi := K(-|u|^{p-1}u) - u = Kv + |v|^{r-1}v$. We would hope to get $\psi = 0$ but all we get from $(**)$ is $\psi \in N$. Then Lemma 9 from the lecture implies that u is a generalized solution of (2) with $T = 2\pi$ (and Brézis, Coron and Nirenberg in fact prove that u is even a classical solution).

We finish this section by giving the definition of a generalized solution of (2) with $T = 2\pi$.

Definition 7 (generalized solution)**.** *Let $f : (0,\pi) \times \mathbb{R} \to \mathbb{R}$ be 2π-periodic in t, $f \in L^1(\Omega)$. A function $u : (0,\pi) \times \mathbb{R} \to \mathbb{R}$, which is 2π-periodic in t with $u \in L^1(\Omega)$ is called a generalized solution of* (3) *if*

$$\int_\Omega u(\phi_{tt} - \phi_{xx})\, d(x,t) = \int_\Omega f\phi\, d(x,t)$$

for all $\phi \in C^2(\overline{\Omega})$, $\phi(0,t) = \phi(\pi,t) = 0$, $\phi(x,t+2\pi) = \phi(x,t)$ for all $(x,t) \in [0,\pi] \times \mathbb{R}$.

4.2. **Standing waves via the implicit function theorem.**

$$(9) \qquad \begin{cases} u_{tt} - u_{xx} + mu &= \pm u^3 \quad \text{on } (0,\pi) \times \mathbb{R}, \\ u(0,t) = u(\pi,t) &= 0 \quad \text{for all } t \in \mathbb{R}, \\ u(x,t+T) &= u(x,t) \quad \text{for all } (x,t) \in (0,\pi) \times \mathbb{R}, \\ u(x,-t) &= u(x,t) \quad \text{for all } (x,t) \in (0,\pi) \times \mathbb{R}. \end{cases}$$

Remark. The free vibrations of the equation are given by $\sin lx \cos\sqrt{l^2+m}t$. In the following we consider in particular the fundamental frequency $\omega_1 := \sqrt{1+m}$.

Definition. *Let $\gamma > 0$. Consider the set*

$$S_\gamma^m = \{\omega \in \mathbb{R} : |\omega j - \sqrt{l^2+m}| \geq \frac{\gamma}{j} \text{ for all } j, l \in \mathbb{N}, l \neq 1\}$$

of all γ-nonresonant frequencies ω.

Assumption. *There exists countably many values $m > 0$ with the property that for each such m there exists a value $\gamma = \gamma(m) > 0$ such that the set S_γ^m contains sequences $(\omega_k^-)_{k\in\mathbb{N}}, (\omega_k^+)_{k\in\mathbb{N}}$ with $\omega_k^- < \sqrt{1+m} < \omega_k^+$ and $\lim_{k\to\infty}\omega_k^\pm = \sqrt{1+m}$.*

Theorem 16. *Let $m > 0$ and $\omega_k^\pm \in S_\gamma^m$ be as in the Assumption. For sufficiently large k there exists a $\frac{2\pi}{\omega_k^\pm}$-periodic non-trivial classical solution u_k of* (9) *with $\|u_k\|_{H^2} \to 0$ as $k \to \infty$.*

Remark. For k large enough, the function u_k is not stationary, i.e., $\|\partial_t u_k\| \neq 0$.

4.3. **Standing waves via fixed point methods.**

$$
(11) \qquad \begin{cases} u_{tt} - u_{xx} &= f(x,t,u) \quad \text{on } (0,\pi)\times\mathbb{R}, \\ u(0,t) = u(\pi,t) &= 0 \quad \text{for all } t\in\mathbb{R}, \\ u(x,t+\pi) &= u(x,t) \quad \text{for all } (x,t)\in(0,\pi)\times\mathbb{R}, \\ u(x,-t) &= u(x,t) \quad \text{for all } (x,t)\in(0,\pi)\times\mathbb{R}, \\ u(\frac{\pi}{2}-x,t) &= u(\frac{\pi}{2}+x,t) \quad \text{for all } (x,t)\in(0,\frac{\pi}{2})\times\mathbb{R}. \end{cases}
$$

Remark. The fact that we require π-periodicity and evenness in time as well as evenness in space around $\frac{\pi}{2}$ has the following effect on the operator $L=\partial_t^2-\partial_x^2$ together with the boundary/periodicity/evenness-conditions: its null-space consists only of $\{0\}$ and the set of eigenvalues is $\Lambda=\{(2k+1)^2-(2l)^2 : k,l\in\mathbb{N}_0\}$.

Remark. $\Lambda = 4\mathbb{Z}+1$. To see this notice that $\lambda\in\Lambda$ is of the form $\lambda=(2k+1)^2-(2l)^2 = 4(k^2-l^2+k)+1$. This explains the inclusion $\Lambda\subset 4\mathbb{Z}+1$. To see equality take first $k=l$. This shows that $\Lambda\supset 4\mathbb{N}_0+1$. Taking $l=k+1$ shows that $\Lambda\supset -4\mathbb{N}+1$. Hence we may write $\Lambda=\{\lambda_i=4i+1 : i\in\mathbb{Z}\}$ with an increasing sequence of eigenvalues $\ldots<\lambda_{i-1}<\lambda_i<\lambda_{i+1}<\ldots$. Note however that in the sequence $(\lambda_i)_{i\in\mathbb{Z}}$ multiplicities are not counted.

We make us of the following assumption on the nonlinearity f:

$$
(F) \quad \begin{cases} f : \begin{cases} [0,\pi]\times\mathbb{R}\times\mathbb{R} & \to \mathbb{R}, \\ (x,t,s) & \mapsto f(x,t,s) \end{cases} \\ \text{is } \pi\text{-periodic in } t, \text{ even in } x \text{ around } \frac{\pi}{2}, \text{ even in } t \\ \forall M>0\; f, \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial t}, \dfrac{\partial f}{\partial s} : [0,\pi]\times\mathbb{R}\times[-M,M]\to\mathbb{R} \text{ is } \alpha\text{-Hölder cont. in } (x,t,s) \end{cases}
$$

Remark. Assumption (F) implicitely contains that the derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}$ exist.

Theorem 17. *Assume (F). If furthermore f is bounded then there exists a classical solution of* (11).

Theorem 18. *Assume (F). If furthermore for some $i\in\mathbb{Z}$*

$$
4i+1=\lambda_i < \inf_{[0,\pi]\times\mathbb{R}^2}\frac{\partial f}{\partial s} \le \sup_{[0,\pi]\times\mathbb{R}^2}\frac{\partial f}{\partial s} < \lambda_{i+1} = 4i+5
$$

then there exists a unique classical solution of (11).

References

For Section 2 (Lyapunov's center theorem):

1. Amann, Herbert: Ordinary differential equations. An introduction to nonlinear analysis. de Gruyter Studies in Mathematics, 13. Walter de Gruyter & Co., Berlin, 1990. xiv+458 pp.

For Section 3:

1. Levandosky, Steven: Stability and instability of fourth-order solitary waves. J. Dynam. Differential Equations 10 (1998), no. 1, 151–188.
2. Yue Chen, P.J. McKenna: Traveling waves in a nonlinearly suspended beam: theoretical results and numerical observations. J. Differential Equations 136 (1997), no. 2, 325–355.

For Section 4.1:

1. Struwe, Michael: Variational Methods, Chapter I.6 (Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics 34. Springer-Verlag, Berlin, 2008. xx+302).
2. Brézis, Haim; Coron, Jean-Michel; Nirenberg, Louis: Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz. Comm. Pure Appl. Math. 33 (1980), no. 5, 667–684.
3. Lovicarová, Hana: Periodic solutions of a weakly nonlinear wave equation in one dimension. Czechoslovak Math. J. 19 (94) 1969 324–342.

For Section 4.2:

1. Bambusi, Dario: Lyapunov center theorem for some nonlinear PDE's: a simple proof. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 29 (2000), no. 4, 823–837.
2. Kuksin, Sergei B.: Hamiltonian PDEs. With an appendix by Dario Bambusi. Handbook of dynamical systems. Vol. 1B, 1087–1133, Elsevier B. V., Amsterdam, 2006.