Classical Methods for Partial Differential Equations
Lecture Notes – Winter Semester 2023/24

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Classical Methods for Partial Differential Equations are well understood and there exist several standard textbooks like:

- L. C. Evans: Partial Differential Equations
- G. B. Folland: Introduction to Partial Differential Equations
- D. Gilbarg, N. S. Trudinger: Elliptic Partial Differential Equations of Second Order
- D. Hilbert, R. Courant: Methods for Mathematical Physics, Vol. I & II
- F. John: Partial Differential Equations
- J. Jost: Partial Differential Equations
- R. Leis: Vorlesungen über partielle Differentialgleichungen zweiter Ordnung
- F. Treves: Linear Partial Differential Equations with Constant Coefficients
- M. Giaquinta, L. Martinazzi: An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs
- D. Breit, F. Gmeineder: A Course on Function Spaces

With a rich history, the results presented below are not new, but always with a personal touch from the presenter. Furthermore, these notes have also been inspired from the pde lecture by Ulrich Dierkes from the University of Duisburg-Essen and from some "KIT memory" in the form of notes on the preceding versions of these courses by Wolfgang Reichel and Michael Plum.

These lecture notes are a living document. This document is typeset in \LaTeX{}, all pictures are drawn using PStricks. Comments, corrections and remarks are appreciated and can be send via e-mail to: peter.lewintan@kit.edu

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1 Notation

For $n \in \mathbb{N} := \{1, 2, \ldots\}$ we denote by $\mathbb{R}^n$ the $n$-dimensional Euclidean space. Its elements $x \in \mathbb{R}^n$ are seen as column vectors:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

with transpose $x^T = (x_1, \ldots, x_n)$ as a row vector. The basis vectors of $\mathbb{R}^n$ are denoted by

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\text{th entry}, \quad \text{with } j = 1, \ldots, n.$$

For the scalar product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ we have $\langle x, y \rangle := \sum_{j=1}^{n} x_j y_j$ for all $x, y \in \mathbb{R}^n$, and $x, y \in \mathbb{R}^n \setminus \{0\}$ are called perpendicular iff ("if and only if") $\langle x, y \rangle = 0$.

For the (Euclidean) norm $|\cdot| : \mathbb{R}^n \to [0, \infty)$ we have

$$|x| := \sqrt{\langle x, x \rangle} \text{ for all } x \in \mathbb{R}^n.$$

In $\mathbb{R}^3$ the three basic defining properties of the vector product (or cross product) $\cdot \times \cdot : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ are:

i) the linearity in both arguments,

ii) that the vector $a \times b$ is perpendicular to both $a, b \in \mathbb{R}^3$ (and thus belongs to the same space, see Fig. 1) and

iii) that its length has the same absolute value as the area of the parallelogram spanned by $a$ and $b$:

$$|a \times b|^2 = |a|^2 |b|^2 - \langle a, b \rangle^2 \quad \forall a, b \in \mathbb{R}^3. \quad (1.1)$$

Exercise

Conclude from the area property (1.1) the anti-commutativity of the vector product, namely

$$a \times b = -b \times a \quad \forall a, b \in \mathbb{R}^3.$$

Since for a fixed vector $a \in \mathbb{R}^3$ the map $a \times \cdot : \mathbb{R}^3 \to \mathbb{R}^3$ is linear there exists a matrix $A \in \mathbb{R}^{3\times 3}$ such that

$$a \times b = Ab \quad \forall b \in \mathbb{R}^3,$$

more precisely we have

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} =: \text{Anti } a, \quad (1.2)$$

i.e. $a \times b = (\text{Anti } a) b$. Finally, the dyadic product (or tensor product) $\cdot \otimes \cdot : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{m \times n}$ is given by

$$a \otimes b := ab^T \quad \forall a \in \mathbb{R}^m, b \in \mathbb{R}^n.$$
In the following we will denote by \( \Omega \subseteq \mathbb{R}^n \) an open and non-empty subset (which is not necessarily bounded). If in addition it is also connected then \( \Omega \) is called a domain. We will denote by \( \partial \Omega \) the boundary of \( \Omega \) and by \( \overline{\Omega} := \Omega \cup \partial \Omega \) the closure of \( \Omega \).

We say a function \( u : \Omega \to \mathbb{R}^m \) is scalar valued in case \( m = 1 \) and vector valued if \( m \geq 2 \). With \( x \in \Omega \) we call the following limit the \( j \)th partial derivative if it exists

\[
\lim_{h \to 0} \frac{1}{h} (u(x + h e_j) - u(x)) =: \frac{\partial u}{\partial x_j}(x) = u_{x_j}(x) = \begin{pmatrix} \frac{\partial u_1}{\partial x_j}(x) \\ \vdots \\ \frac{\partial u_m}{\partial x_j}(x) \end{pmatrix}.
\]

If for all \( j \in \{1, \ldots, n\} \) we have \( \frac{\partial u}{\partial x_j} \in C^0(\Omega; \mathbb{R}^m) \) then \( u \) is called to be continuously differentiable which is denoted by \( u \in C^1(\Omega; \mathbb{R}^m) \).

Arranging these columns gives the derivative (or gradient or Jacobi matrix):

\[
\frac{\partial u}{\partial x_1} \cdots \frac{\partial u}{\partial x_n} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_m}{\partial x_1} & \cdots & \frac{\partial u_m}{\partial x_n} \end{pmatrix} =: Du = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} \\ \vdots \\ \frac{\partial u_m}{\partial x_1} \end{pmatrix} \otimes \left( \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \right) = u \otimes \nabla,
\]

with the vector differential operator \( \nabla \).

### Chain rule

For \( f \in C^1(\mathbb{R}^n; \mathbb{R}^m) \) and \( g \in C^1(\mathbb{R}^m; \mathbb{R}^d) \), we have that \( g \circ f \in C^1(\mathbb{R}^n; \mathbb{R}^d) \) with

\[
D(g \circ f) = [(Dg) \circ f] Df.
\]

- In the scalar valued case \( m = 1 \) we distinguish between the row

\[
Du = \begin{pmatrix} \frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u}{\partial x_n} \end{pmatrix}
\]

and the column

\[
\nabla u = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{pmatrix} = (Du)^\top,
\]

and call the latter the gradient of \( u \).

- In case \( m = n \) the derivative \( Du \in \mathbb{R}^{n \times n} \) is a square matrix field. Thus, of particular interest are

  - its trace

\[
\text{tr} \; Du := \langle Du, E_n \rangle = \frac{\partial u_1}{\partial x_1} + \cdots + \frac{\partial u_m}{\partial x_m} = \sum_{j=1}^n \frac{\partial u_j}{\partial x_j} = \langle u, \nabla \rangle = \langle \nabla, u \rangle
\]

the divergence of \( u \).
- its symmetric part
  \[\text{sym } D u := \frac{1}{2} (D u + (D u)^\top),\] and

- its skew-symmetric part
  \[\text{skew } D u := \frac{1}{2} (D u - (D u)^\top).\]

- In the special case \(m = n = 3\) we moreover consider
  \[\text{curl } u := u \times (-\nabla) = -(\text{Anti } u) \nabla = \nabla \times u = (\text{Anti } \nabla) u\]
  the curl of \(u\) (Ger.: rot as "Rotation").

**Exercise**

Show that for all \(u \in C^1(\mathbb{R}^3; \mathbb{R}^3)\) it holds: \(\text{skew } D u = \frac{1}{2} \text{Anti curl } u\).

Finally, we address higher order derivatives. The Hessian of a scalar valued function is given by

\[D \nabla u = \nabla u \otimes \nabla = \nabla \otimes \nabla u = u \nabla \otimes \nabla,\]

which may also be denoted by \(D^2 u\) in the literature. Since the Hessian is a square matrix field, of special interest is its trace, namely the Laplacian:

\[\text{tr } D \nabla u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} := \Delta u = |\nabla|^2 u,\]

showing that the Laplace operator behaves like a scalar differential operator.

Let us now recall the multi-index notation. \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n\) is called a multi-index of order (or length)
\[|\alpha| := \alpha_1 + \ldots + \alpha_n\] (not Euclidean norm here!)

We define
\[D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u.\]

Furthermore, for all \(k \in \mathbb{N}_0:\)
\[C^k(\Omega; \mathbb{R}^m) := \{u : \Omega \to \mathbb{R}^m : D^\alpha u \text{ is continuous for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k\}\]
and set
\[C^\infty(\Omega; \mathbb{R}^m) := \bigcap_{k \in \mathbb{N}_0} C^k(\Omega; \mathbb{R}^m).\]

The set of all partial derivatives of order \(k\) will be denoted by
\[D^k u := \{D^\alpha u : |\alpha| = k\}.\]

**Schwarz theorem**

For all \(u \in C^2(\mathbb{R}^n; \mathbb{R})\) it holds: \(\text{skew } D \nabla u \equiv 0\).
Exercise

Prove the following identities:
(a) \( \text{div } \nabla u = \Delta u \) for \( u \in C^2(\mathbb{R}^3; \mathbb{R}) \),
(b) \( \text{div } \text{curl } u = 0 \) for \( u \in C^2(\mathbb{R}^3; \mathbb{R}) \),
(c) \( \text{curl } \nabla u = 0 \) for \( u \in C^2(\mathbb{R}^3; \mathbb{R}) \),
(d) \( \Delta u \coloneqq \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{pmatrix} = \nabla \text{div } u - \text{curl } \text{curl } u \) for \( u \in C^2(\mathbb{R}^3; \mathbb{R}) \),
(e) \( \text{div}(v \times w) = \langle w, \text{curl } v \rangle - \langle v, \text{curl } w \rangle \) for \( v, w \in C^1(\mathbb{R}^3; \mathbb{R}^3) \).

Remark 1.1. Sometimes we employ a subscript attached to the differential symbol, e.g.: for \( u \in C^1((0, \infty) \times \mathbb{R}^3; \mathbb{R} \) with \((t, \mathbf{x}) \in (0, \infty) \times \mathbb{R}^3 \) we have
\[
\text{div}_x u = \partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3 \quad \text{without any } \partial_t u.
\]
Moreover, the support of a function \( u : \Omega \to \mathbb{R}^m \) is defined by
\[
\text{supp } u \coloneqq \{ x \in \Omega : u(x) \neq 0 \}
\]
and denote by a subscript “c” attached to function spaces the subset of compactly supported functions:
Let \( k \in \mathbb{N}_0 \cup \{ \infty \} \), then
\[
C_c^k(\Omega; \mathbb{R}^m) \coloneqq \{ u \in C^k(\Omega; \mathbb{R}^m) : \text{supp } u \subset \subset \Omega \},
\]
whereby “\( \subset \subset \) means “has compact closure”, more precisely
\[
V \subset \subset \Omega \quad \iff \quad V \subset \overline{V} (\text{compact}) \subset \Omega.
\]

2 Partial differential equations

PDEs are equations involving an unknown functions of two or more variables and certain of its partial derivatives.

Definition. We call an expression of the form
\[
F(D^k u(x), D^{k-1} u(x), \ldots, Du(x), u(x), x) = 0, \quad x \in \Omega
\]
(PDE)
a \textit{kth order partial differential equation}, where
\[
F : \mathbb{R}^{m \times n^k} \times \mathbb{R}^{m \times n^{k-1}} \times \ldots \times \mathbb{R}^{m \times n} \times \mathbb{R}^m \times \Omega \to \mathbb{R}
\]
is given, and \( u : \Omega \to \mathbb{R}^m \) is the unknown.

If \( F \) is vector valued, then (PDE) is also called system of partial differential equations.

Definition. (a) (PDE) is called linear if it is of the form
\[
\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x) = f(x)
\]
with given functions \( a_\alpha \) (so called coefficients, for \( |\alpha| \leq k \)) and \( f \). If \( f \equiv 0 \) then (PDE) is called to be homogeneous.

(b) In the nonlinear case, we highlight the following special cases

(i) (PDE) is called semilinear if it is of the form
\[
\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u(x) + \tilde{F}(D^{k-1} u(x), \ldots, Du(x), u(x), x) = 0.
\]

(ii) (PDE) is called quasilinear if it is of the form
\[
\sum_{|\alpha| = k} a_\alpha(D^{k-1} u(x), \ldots, Du(x), u(x), x) D^\alpha u(x) + \tilde{F}(D^{k-1} u(x), \ldots, Du(x), u(x), x) = 0.
\]
2.1 Examples

2.1.1 Linear equations

1) Laplace’s equation for $u \in C^2(\Omega; \mathbb{R})$:

$$\Delta u = 0 \quad \text{in } \Omega.$$ 

2) Transport equation for $u \in C^1((0, T) \times \Omega; \mathbb{R}), T > 0, b \in \mathbb{R}^n$ fixed:

$$u_t + \langle b, \nabla_x u \rangle = 0 \quad \text{in } (0, T) \times \Omega.$$ 

3) Heat (or diffusion) equation for $u \in C^2_1((0, T) \times \Omega; \mathbb{R}) = C^1((0, T); C^2(\Omega; \mathbb{R}))$:

$$u_t - \Delta_x u = 0 \quad \text{in } (0, T) \times \Omega.$$ 

4) Wave equation for $u \in C^2((0, T) \times \Omega; \mathbb{R})$:

$$u_{tt} - \Delta_x u = 0 \quad \text{in } (0, T) \times \Omega.$$ 

2.1.2 Nonlinear equations

1) Eikonal equation for $u \in C^1(\Omega; \mathbb{R})$:

$$|\nabla u| = 1.$$ 

2) Nonlinear Poisson equation for $u \in C^2(\Omega; \mathbb{R})$:

$$-\Delta u = f(u).$$ 

3) $p$-Laplace equation for $u \in C^2(\Omega; \mathbb{R}), p > 2$ or $1 < p < 2$:

$$\text{div}(|\nabla u|^{p-2} \nabla u) = 0.$$ 

4) Minimal surface equation for $u \in C^2(\Omega; \mathbb{R})$:

$$\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0.$$ \hspace{1cm} (MSE) 

5) Monge-Ampère equation for $u \in C^2(\Omega; \mathbb{R})$:

$$\det(D\nabla u) = f.$$ 

6) Reaction-diffusion equation for $u \in C^2_1((0, T) \times \Omega; \mathbb{R})$:

$$u_t - \Delta_x u = f(u).$$ 

7) Nonlinear wave equation $u \in C^2((0, T) \times \Omega; \mathbb{R})$:

$$u_{tt} - \Delta_x u = f(u).$$
2.1.3 Linear systems

1) Cauchy-Riemann equations on $\Omega \subseteq \mathbb{R}^2$ for $u, v \in C^1(\Omega; \mathbb{R})$:

\[
\begin{cases}
    u_x = v_y, \\
    u_y = -v_x.
\end{cases}
\] (C.-R.)

2) Maxwell’s equations on $(0, T) \times \Omega \subseteq \mathbb{R}^4$ for $E, B \in C^1((0, T) \times \Omega; \mathbb{R}^3)$

\[
\begin{cases}
    \text{curl}_x B = E_t, \\
    \text{curl}_x E = -B_t, \\
    \text{div}_x B = 0, \\
    \text{div}_x E = 0.
\end{cases}
\]

2.1.4 Nonlinear systems

1) Incompressible Navier-Stokes equations for $u \in C^2_1((0, T) \times \Omega; \mathbb{R}^n)$:

\[
\begin{cases}
    u_t + (u, \nabla_x)u - \Delta_x u = -\nabla_x p, \\
    \text{div}_x u = 0.
\end{cases}
\]

Exercise

Show that the minimal surface equation (MSE) is a quasilinear partial differential equation of 2nd order.

2.2 Classification of linear PDEs of 2nd order

Consider a linear PDE of 2nd order

\[ Lu = f \quad \text{in } \Omega \] (2.1)

with the linear 2nd order differential operator $L = \sum_{|\alpha|=0}^2 a_{\alpha}(x) \frac{\partial|\alpha|}{\partial x^\alpha}$ and multi-index $\alpha \in \mathbb{N}_0^n$.

For $x \in \Omega \subseteq \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$ the symbol of $L$ is given by

\[ \sigma_x(L)(\xi) := \sum_{|\alpha|=0}^2 a_{\alpha}(x)\xi^\alpha, \]

i.e., it is a 2nd order polynomial/quadratic form.

Example 2.1. $L = \Delta = |\nabla|^2$ then $\sigma_x(L)(\xi) = |\xi|^2$ for all $x \in \Omega$ (replace $\nabla \leftrightarrow \xi$).

Definition. The linear PDE of 2nd order (2.1) is called

- **elliptic** $\iff \sigma_x(L)$ is positive definite or negative definite on $\mathbb{R}^n$ for all $x \in \Omega$
  (i.e. all eigenvalues are positive or all eigenvalues are negative, we consider only 2nd order terms)
  
  **example**: $L = \Delta$,

- **parabolic** $\iff \sigma_x(L)$ is positive or negative semidefinite on $\mathbb{R}^n$ for all $x \in \Omega$
  (i.e. vanishes in some directions, has eigenvalue 0, we consider only 2nd order terms)
  
  **example**: $L = \frac{\partial}{\partial t} - \Delta_x$ in $(0, T) \times \Omega \subseteq \mathbb{R}^{n+1}$:
  \[ \sigma_{(t,x)}(L)(\xi) = \xi_1 - (\xi_2^2 + \ldots + \xi_{n+1}^2) \text{ with } \xi \in \mathbb{R}^{n+1}, \]
  \( \xi_1 \) does not appear here

- **hyperbolic** $\iff \sigma_x(L)$ is indefinite on $\mathbb{R}^n$ for all $x \in \Omega$ and has only one positive (resp. negative)
  and all remaining negative (resp. positive) eigenvalues (we consider only 2nd order terms)
  
  **example**: $L = \frac{\partial^2}{\partial t^2} - \Delta_x$ in $(0, T) \times \Omega \subseteq \mathbb{R}^{n+1}$:
  \[ \sigma_{(t,x)}(L)(\xi) = \xi_1^2 - (\xi_2^2 + \ldots + \xi_{n+1}^2) \text{ with } \xi \in \mathbb{R}^{n+1}. \]
2.3 Boundary conditions

Often there are boundary conditions which are added to the differential equation:

a) Dirichlet boundary conditions: u is prescribed on the boundary:

\[ u = \phi \quad \text{on } \partial \Omega, \quad \phi : \partial \Omega \to \mathbb{R}^m \text{ continuous} \]

b) Neumann boundary conditions: the normal derivative of u is prescribed on the boundary:

\[ \frac{\partial u}{\partial \nu} = (Du)\nu = \phi \quad \text{on } \partial \Omega, \]

where \( \nu = \nu(x) \) is the unit outer normal vector field along \( \partial \Omega \) (need some regularity assumptions on the boundary, e.g. Lipschitz)

c) If Dirichlet and Neumann boundary conditions occur on different parts of the boundary we call it mixed boundary conditions.

**Example 2.2.** For \( \Omega \) (bounded domain) \( \subset \mathbb{R}^n \) and \( u \in C^2(\Omega; \mathbb{R}) \) consider

\[
\begin{cases}
\Delta u = 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega.
\end{cases}
\]

A multiplication of the Laplace equation with \( u \) and an integration over \( \Omega \) gives

\[
\int_{\Omega} u \Delta u \, dx = 0.
\]

Integrating by parts we obtain

\[
- \int_{\Omega} \langle \nabla u, \nabla u \rangle \, dx + \int_{\partial \Omega} u \langle \nabla u, \nu \rangle \, dA = 0,
\]

where \( dA \) denotes the surface measure on \( \partial \Omega \). Thus, using the boundary condition \( u = 0 \) on \( \partial \Omega \) the second term vanishes and we obtain

\[
\int_{\Omega} |\nabla u|^2 \, dx = 0 \quad \Rightarrow \quad |\nabla u|^2 = 0 \quad \Rightarrow \quad u \equiv \text{const}.
\]

Since \( u \) is continuous up to the boundary, the vanishing of the boundary data implies the vanishing of \( u \):

\[
u = 0.
\]

**Exercise**

Conclude from example 2.2 the uniqueness of \( C^2 \)-solutions of Laplace’s equation with prescribed boundary data.

**Remark 2.1.** The presence of boundary data does not guarantee the solvability of a boundary value problem (BVP = PDE & BC). One can show that

\[
\begin{cases}
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \quad \text{(MSE)} \text{ in } \Omega \subset \mathbb{R}^2, \\
u = \varphi \quad \text{on } \partial \Omega, \varphi \in C^0,
\end{cases}
\]

is solvable if and only if \( \Omega \) is convex.
2.4 Euler-Lagrange equations

Some differential equations arise from variational problems: Consider the variational integral

$$\mathcal{F}(u) := \int_\Omega F(\nabla u(x), u(x), x) \, dx$$

for $u : \Omega \to \mathbb{R}$ and an integrand $F : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}, (p, z, x) \mapsto F(p, z, x)$.

Example:

- $F(p, z, x) = \sqrt{1 + |p|^2} \Rightarrow$ area functional
- $F(p, z, x) = \frac{1}{2} |p|^2 \Rightarrow$ Dirichlet functional

Assumption: $u \in C^2(\overline{\Omega}; \mathbb{R})$ minimizes $\mathcal{F}(\cdot)$ among the class

$$\mathcal{C}(u) := \{ v \in C^2(\overline{\Omega}; \mathbb{R}), v = u \text{ on } \partial \Omega \},$$

i.e. $\mathcal{F}(u) \leq \mathcal{F}(v)$ for all $v \in \mathcal{C}(u)$. In particular we have

$$\mathcal{F}(u) \leq \mathcal{F}(u + \varepsilon \varphi) \quad \forall \ |\varepsilon| < 1, \forall \varphi \in C^\infty_c(\Omega; \mathbb{R}),$$

since $u + \varepsilon \varphi \in \mathcal{C}(u)$. Setting $f(\varepsilon) := \mathcal{F}(u + \varepsilon \varphi)$ the last inequality reads

$$f(0) \leq f(\varepsilon) \quad \text{with admissible } f : (-1, 1) \to \mathbb{R}.$$  

This can be rewritten as

$$\left. \frac{df}{d\varepsilon} \right|_{\varepsilon=0} = f'(0) = 0$$

$$= \frac{d}{d\varepsilon} \mathcal{F}(u + \varepsilon \varphi) \bigg|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon} \left\{ \int_\Omega F(\nabla u + \varepsilon \nabla \varphi, u + \varepsilon \varphi, x) \, dx \right\} \bigg|_{\varepsilon=0}$$

meaning that we have to differentiate a parameter dependent integral interchanging two limites. To this aim, we add the following assumptions: $\Omega$ bounded domain and $F \in C^2(\mathbb{R}^n \times \mathbb{R} \times \overline{\Omega}; \mathbb{R})$. Thus, differentiation under the integral is allowed due to smoothness and compactness reasons:

$$0 = \int_\Omega \frac{d}{d\varepsilon} F(\nabla u + \varepsilon \nabla \varphi, u + \varepsilon \varphi, x) \bigg|_{\varepsilon=0} \, dx$$

chain rule

$$= \int_\Omega \left\{ \langle \nabla_p F(\nabla u, u, x), \nabla_x \varphi \rangle + \frac{\partial}{\partial z} F(\nabla u, u, x) \varphi \right\} \, dx$$

integration by parts

$$\varphi = 0 \text{ on } \partial \Omega$$

$$\forall \varphi \in C^\infty_c(\Omega; \mathbb{R}).$$

(2.2)

**Fundamental lemma of the calculus of variations**

Assume that $g \in C^0(\Omega; \mathbb{R})$ satisfies $\int_\Omega g \varphi \, dx = 0$ for all $\varphi \in C^\infty_c(\Omega; \mathbb{R})$. Then $g \equiv 0$ in $\Omega$.  

Figure 3: Indicated is a graph of a function $u$ with fixed boundary values and in green and in red two compactly supported variations of $u$.

\[ \left. \frac{df}{d\varepsilon} \right|_{\varepsilon=0} = f'(0) = 0 \]

\[ = \frac{d}{d\varepsilon} \mathcal{F}(u + \varepsilon \varphi) \bigg|_{\varepsilon=0} \]

\[ = \frac{d}{d\varepsilon} \left\{ \int_\Omega F(\nabla u + \varepsilon \nabla \varphi, u + \varepsilon \varphi, x) \, dx \right\} \bigg|_{\varepsilon=0} \]

meaning that we have to differentiate a parameter dependent integral interchanging two limites. To this aim, we add the following assumptions: $\Omega$ bounded domain and $F \in C^2(\mathbb{R}^n \times \mathbb{R} \times \overline{\Omega}; \mathbb{R})$. Thus, differentiation under the integral is allowed due to smoothness and compactness reasons:

\[ 0 = \int_\Omega \frac{d}{d\varepsilon} F(\nabla u + \varepsilon \nabla \varphi, u + \varepsilon \varphi, x) \bigg|_{\varepsilon=0} \, dx \]

chain rule

\[ \int_\Omega \left\{ \langle \nabla_p F(\nabla u, u, x), \nabla_x \varphi \rangle + \frac{\partial}{\partial z} F(\nabla u, u, x) \varphi \right\} \, dx \]

integration by parts

\[ \varphi = 0 \text{ on } \partial \Omega \]

\[ \forall \varphi \in C^\infty_c(\Omega; \mathbb{R}). \]

(2.2)

**Fundamental lemma of the calculus of variations**

Assume that $g \in C^0(\Omega; \mathbb{R})$ satisfies $\int_\Omega g \varphi \, dx = 0$ for all $\varphi \in C^\infty_c(\Omega; \mathbb{R})$. Then $g \equiv 0$ in $\Omega$.  

\[ 0 = \int_\Omega \frac{d}{d\varepsilon} F(\nabla u + \varepsilon \nabla \varphi, u + \varepsilon \varphi, x) \bigg|_{\varepsilon=0} \, dx \]

chain rule

\[ = \int_\Omega \left\{ \langle \nabla_p F(\nabla u, u, x), \nabla_x \varphi \rangle + \frac{\partial}{\partial z} F(\nabla u, u, x) \varphi \right\} \, dx \]

integration by parts

\[ \varphi = 0 \text{ on } \partial \Omega \]

\[ \forall \varphi \in C^\infty_c(\Omega; \mathbb{R}). \]
Thus, by the fundamental lemma of the calculus of variations we conclude from (2.2) the validity of

\[
\text{div}_x \left( \nabla_p F(\nabla u, u, x) \right) = \frac{\partial}{\partial z} F(\nabla u, u, x)
\]

in \( \Omega \). (E.-L.)

Equation (E.-L.) is called the Euler-Lagrange equation of the variational integral \( \mathcal{F}(\cdot) \). This equation is a (quasilinear) PDE of 2nd order. Indeed, differentiating the left hand side of (E.-L.) gives (applying the chain rule):

\[
\left\langle (\nabla_p \nabla F)(\nabla u, u, x), D\nabla u \right\rangle + \left( \left( \frac{\partial}{\partial z} \nabla_p F \right)(\nabla u, u, x) \right) \right) + \left( \text{div}_x \nabla_p F(\nabla u, u, x) \right) = \frac{\partial}{\partial z} F(\nabla u, u, x).
\]

Summarizing, we have shown

**Theorem 2.2.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, \( F \in C^2(\mathbb{R}^n \times \mathbb{R} \times \Omega; \mathbb{R}) \) and \( u \in C^2(\overline{\Omega}; \mathbb{R}) \) be a minimizer of the variational integral

\[
\mathcal{F}(u) = \int_{\Omega} F(\nabla u(x), u(x), x) \, dx
\]

with respect to its own boundary values. Then \( u \) fulfills the Euler-Lagrange equation (E.-L.).

**Example 2.3.**
- Considering the Dirichlet energy

\[
\mathcal{D}(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx
\]

we have \( F(p, z, x) = \frac{1}{2} |p|^2 \). Since \( \nabla_p F(p, z, x) = p, \frac{\partial}{\partial z} F(p, z, x) = 0 \) and \( \nabla_x F(p, z, x) = 0 \) we obtain as Euler-Lagrange equation the Laplace equation:

\[
0 = \text{div}_x \nabla_p F(\nabla u, u, x) = \text{div}_x \nabla_x u = \Delta u.
\]

- (Exercise:) The minimal surface equation (MSE) is the Euler-Lagrange equation of the area functional

\[
\mathcal{A}(u) := \int_{\Omega} \sqrt{1 + |\nabla u|^2} \, dx.
\]

### 3 Laplace and Poisson equation

#### 3.1 Harmonic functions

**Definition.** Let \( \Omega \subset \mathbb{R}^n \) be open. A function \( u \in C^2(\Omega; \mathbb{R}) \) is called harmonic iff \( \Delta u \equiv 0 \) in \( \Omega \).

**Examples**
- in \( \Omega = \mathbb{R}^n \): \( u(x) = \langle a, x \rangle + \beta, \quad a \in \mathbb{R}^n, \beta \in \mathbb{R} \) fixed,
- in \( \Omega = \mathbb{R}^2 \): \( u(x, y) = x^2 - y^2 \),
- in \( \Omega = \mathbb{R}^3 \): \( u(x, y, z) = xy + z \).

**Remark 3.1.** If \( \Omega \subseteq \mathbb{R}^2 \) and \( u, v \in C^2(\Omega; \mathbb{R}) \) satisfy the Cauchy-Riemann equations

\[
\begin{cases}
u_x = v_y, \\
u_y = -v_x,
\end{cases}
\]

(C.-R.)

then \( u \) and \( v \) are harmonic functions. Indeed,

\[
u_{xx} + u_{yy} = v_{yx} - v_{xy} \equiv 0 \quad \text{as well as} \quad v_{xx} + u_{yy} = -u_{yx} + u_{xy} \equiv 0.
\]

Recall that the Cauchy-Riemann equations (C.-R.) are fulfilled iff \( f(z) = f(x + iy) = u(x, y) + iv(x, y) \) is a holomorphic function in \( z = x + iy \in \mathbb{C} \). Thus, other examples of harmonic functions are the real/imaginary part of any holomorphic function:

\[
\begin{array}{c|c|c}
\text{function} & u = \text{Re} f & v = \text{Im} f \\
\hline
\frac{z^2}{x^2} & \frac{x^2 - y^2}{2xy} & \frac{e^x}{e^x \cos y} \\
\hline
\frac{z}{e^x} & \frac{e^x \sin y}{e^x} & \text{Log } z = \ln \sqrt{x^2 + y^2} \quad \text{arctan } \frac{x}{y} & \text{take } x > 0.
\end{array}
\]
3.2 Ostrogradsky, Gauß, Green, Stokes

We recall the

**Fundamental theorem of calculus**

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with sufficiently smooth boundary (e.g. Lipschitz). Then for all \( \varphi \in C^1(\Omega; \mathbb{R}) \cap C^0(\overline{\Omega}; \mathbb{R}) \) it holds

\[
\int_{\Omega} \partial_j \varphi \, dx = \int_{\partial \Omega} \nu_j \varphi \, dA \quad \forall \; j = 1, \ldots, n,
\]

where \( \nu \) denotes the outer unit normal field along \( \partial \Omega \).

"interchange \( \partial_j \leftrightarrow \nu_j \)"

This is enough to deduce

1) the divergence theorem (Ostrogradsky / Gauß):

by summing up over all \( j \):

\[
\int_{\Omega} \text{div} \; f \, dx = \int_{\partial \Omega} \langle \nu, f \rangle \, dA \quad \forall \; f \in C^1(\Omega; \mathbb{R}^n) \cap C^0(\overline{\Omega}; \mathbb{R}^n).
\]

2) integration by parts formula:

To this end, consider \( f = \alpha h \) with \( \alpha \in C^1(\Omega; \mathbb{R}) \cap C^0(\overline{\Omega}; \mathbb{R}) \), \( h \in C^1(\Omega; \mathbb{R}^n) \cap C^0(\overline{\Omega}; \mathbb{R}^n) \) and note that by the product rule we have

\[
\text{div} \; f = \langle \nabla \alpha, h \rangle + \alpha \text{div} \; h,
\]

so that we conclude

\[
\int_{\Omega} \langle \nabla \alpha, h \rangle \, dx = - \int_{\Omega} \alpha \text{div} \; h \, dx + \int_{\partial \Omega} \alpha \langle h, \nu \rangle \, dA.
\]

3) Green’s first identity:

Consider a gradient field \( h = \nabla \beta \) with \( \beta \in C^2(\Omega; \mathbb{R}) \cap C^1(\overline{\Omega}; \mathbb{R}) \) then:

\[
\int_{\Omega} \left( \alpha \Delta \beta + \langle \nabla \alpha, \nabla \beta \rangle \right) \, dx = \int_{\partial \Omega} \alpha \langle \nabla \beta, \nu \rangle \, dA.
\]

4) Green’s second identity:

Interchange the roles of \( \alpha \) and \( \beta \) and subtract: We have

\[
\int_{\Omega} \left( \beta \Delta \alpha + \langle \nabla \alpha, \nabla \beta \rangle \right) \, dx = \int_{\partial \Omega} \beta \langle \nabla \alpha, \nu \rangle \, dA,
\]

and, thus, deduce

\[
\int_{\Omega} \left( \alpha \Delta \beta - \beta \Delta \alpha \right) \, dx = \int_{\partial \Omega} \left( \alpha \frac{\partial \beta}{\partial \nu} - \beta \frac{\partial \alpha}{\partial \nu} \right) \, dA,
\]

for all \( \alpha, \beta \in C^2(\Omega; \mathbb{R}) \cap C^1(\overline{\Omega}; \mathbb{R}) \).

5) as special case we also obtain by taking \( f = \nabla u \)

\[
\int_{\Omega} \Delta u \, dx = \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \, dA \quad \forall \; u \in C^2(\Omega; \mathbb{R}) \cap C^1(\overline{\Omega}; \mathbb{R}).
\]

6) also Stokes …
Remark 3.2. Concerning the “origin” of Laplace equation e.g. consider a liquid flow. Then for any volume element $V$ the net flux, i.e. the “fluid passing through surface”, is given by

$$\int_{\partial V} \langle F, \nu \rangle \, dA = \int \text{div} \, F \, dx,$$

whereby $F$ denotes the flux density. Assuming moreover the incompressibility of the liquid means that the amount of liquid inside a closed fixed volume is constant, i.e. the amounts of the flowing in and out are equal, and, hence, the net flux $= 0$. Since this condition is fulfilled for any volume element $V$ we conclude

$$\text{div} \, F \equiv 0.$$

In application, $F$ is often given by a gradient field $\nabla u$ (i.e. $\text{curl} \, F \equiv 0$) which then results in $\Delta u \equiv 0$.

### 3.3 Fundamental solution

One of the principal features of the Laplacian is its spherical symmetry, see Exercise 7: its value is preserved under rotations about any fixed point $a \in \mathbb{R}^n$. Let us search for special solutions of the Laplace equation that are invariant under rotations about $a$, i.e. have the same value at all points at the same distance from $a$. Such solutions would be of the form

$$v(x) = \Psi_a(|x - a|)$$

with $\Psi_a : I \to \mathbb{R}$, $r \mapsto \Psi_a(r)$, and admissible interval $I \subset \mathbb{R}$. We have

$$Dv = D_x (\Psi_a(|x - a|)) \overset{\text{chain rule}}{=} \Psi'_a(|x - a|) D_x |x - a| = \Psi'_a(|x - a|) \frac{(x - a)^T}{|x - a|} \quad \text{(3.1)}$$

and thus

$$\Delta v = \langle \nabla, \nabla \rangle \overset{\text{product rule}}{=} \langle \nabla_x \Psi_a'(|x - a|), \frac{x - a}{|x - a|} \rangle$$

$$= \Psi''_a(|x - a|) \frac{x - a}{|x - a|} - \Psi'_a(|x - a|) \frac{x - a}{|x - a|} + \frac{1}{|x - a|} \langle \nabla_x |x - a|, x - a \rangle$$

$$= \frac{1}{|x - a|^2} \langle \nabla_x |x - a|, x - a \rangle + \frac{n}{|x - a|}$$

$$= \frac{n - 1}{|x - a|}.$$ (3.2)

Meaning that

$$\Delta v \equiv 0 \iff \Psi''_a(r) + \Psi'_a(r) \frac{n - 1}{r} \equiv 0 \text{ with } r = |x - a|$$

$$= \frac{1}{r^{n-1}} \frac{d}{dr} (r^{n-1} \Psi'_a(r))$$

and we only have to solve

$$\frac{d}{dr} (r^{n-1} \Psi'_a(r)) \equiv 0 \text{ in } I.$$

Since $I$ was an interval, we obtain

$$r^{n-1} \Psi'_a(r) \equiv c \quad \text{(const)} \iff \Psi'_a(r) = \frac{c}{r^{n-1}} \implies \Psi_a(r) = \begin{cases} c \ln r, & \text{when } n = 2, \\ \frac{c r^{2-n}}{2-n}, & \text{when } n \geq 3, \end{cases}$$

without adding a trivial constant.
Proposition 3.3 (Properties of $\Psi_a$). Let $a \in \mathbb{R}^n$, then with $v(x) := \Psi_a(|x - a|)$ we have

i) $v \in C^\infty(\mathbb{R}^n \setminus \{a\}; \mathbb{R})$, $\Psi_a \in C^\infty((0, +\infty); \mathbb{R})$ and $v$ solves the Laplace equation $\Delta v \equiv 0$ in $\mathbb{R}^n \setminus \{a\}$,

ii) $\Psi_a : (0, +\infty) \to \mathbb{R}$ is strictly monotone,

iii) $\nabla_x (\Psi_a(|x - a|)) = c \frac{x - a}{|x - a|^n}$ \Rightarrow $|\nabla_x (\Psi_a(|x - a|))| \leq c |x - a|^{1-n}$,

iv) $|D_x \nabla_x (\Psi_a(|x - a|))| \leq \tilde{c} |x - a|^{-n}$.

v) Let $R > 0$ and $B_R(a) := \{x \in \mathbb{R}^n : |x - a| < R\}$ denote the open ball of radius $R$ and midpoint $a$. Then for all $x \in \partial B_R(a)$ it holds

\[
\frac{\partial v}{\partial \nu}(x) = \frac{\partial (\Psi_a(|x - a|))}{\partial \nu} = \Psi_a'(|x - a|) = c \frac{a}{|x - a|^{n-1}} = \frac{c}{R^{n-1}}.
\]

vi) Although $\nu$ has a singularity in $a$ it is locally integrable, i.e. $v \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R})$, more explicitly: for all $R > 0$ we have $v \in L^1(B_R(a); \mathbb{R})$, i.e.

\[
\int_{B_R(a)} |\Psi_a(|x - a|)| \, dx < \infty.
\]

Proof. Ad i), ii) and iii) see the calculations before the statement of the proposition.

Ad iv): We have

\[
D_x \nabla_x (\Psi_a(|x - a|)) = \frac{c}{\pi} \frac{x - a}{|x - a|^n},
\]

\[
\Rightarrow (\Psi_a(|x - a|)) = c \left\{ \frac{1}{|x - a|^n} D_x (x - a) + (x - a) D_x \frac{1}{|x - a|^n} \right\}
\]

\[
= c \left\{ \frac{1}{|x - a|^n} E_n - \frac{n}{|x - a|^{n+1}} (x - a) D_x |x - a| \right\}
\]

\[
= c \left\{ \frac{1}{|x - a|^n} E_n - \frac{n}{|x - a|^{n+1}} (x - a) (x - a)^\top \right\}
\]

\[
= c \frac{1}{|x - a|^n} \left\{ E_n - n (x - a) \otimes (x - a) \right\}
\]

(3.3)

Ad v): Since the unit outer normal vector field on $\partial B_R(a)$ is given by $\frac{x - a}{|x - a|}$ we obtain

\[
\frac{\partial v}{\partial \nu}(x) = \langle \nabla_x v, \nu(x) \rangle = \langle \nabla_x \Psi_a(|x - a|), \frac{x - a}{|x - a|} \rangle
\]

\[
= c \langle \frac{x - a}{|x - a|^n}, \frac{x - a}{|x - a|} \rangle = c \frac{1}{|x - a|^{n-1}} = \Psi_a'(|x - a|).
\]

Ad vi): We denote by $V_n := \mathcal{L}^n(B_1(0))$ the $n$-dimensional volume of the unit ball\(^1\). Then by the radial symmetry of $\Psi_a(|x - a|)$ we obtain

\[
\int_{B_R(a)} |\Psi_a(|x - a|)| \, dx = n V_n \int_0^R |\Psi_a(r)| r^{n-1} \, dr = \begin{cases} |c| \int_0^R \frac{r}{\ln r} \, dr, & n = 2, \\
|c| \frac{n V_n}{n - 2} \int_0^R r \, dr, & n \geq 3,
\end{cases}
\]

and these integrals exist (since the integrand is continuous-ly extendable in 0).

\(^1\)more precisely we have $V_1 = 2, V_2 = \pi, V_3 = \frac{4}{3} \pi, \ldots, V_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}$
Choice of the constant $c$:

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary. Since above $v$ is singular at $x = a \in \Omega$ we cut out from $\Omega$ a closed ball $B_\varepsilon(a)$ with sufficiently small radius $\varepsilon > 0$. Note that

$$\partial \left( \Omega \setminus B_\varepsilon(a) \right) \supset \overline{B_\varepsilon(a)} \cup \partial B_\varepsilon(a),$$

and $v \in C^\infty(\overline{\Omega \setminus B_\varepsilon(a)}; \mathbb{R})$ with $\Delta v \equiv 0$ in $\Omega \setminus B_\varepsilon(a)$. For an arbitrary function $u \in C^2(\Omega; \mathbb{R}) \cap C^1(\overline{\Omega}; \mathbb{R})$ and above $v$ we apply Green’s second identity on $\Omega \setminus B_\varepsilon(a)$:

$$\int_{\Omega \setminus B_\varepsilon(a)} v \Delta u \, dx = \int_{\partial \Omega} \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) \, dA + \int_{\partial B_\varepsilon(a)} \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) \, dA. \quad (3.5)$$

Starting with the last integral on the right hand side:

$$- \int_{\partial B_\varepsilon(a)} u \frac{\partial v}{\partial n} \, dA \stackrel{\text{Prop. 3.3 vi)}}{=} \frac{c}{\varepsilon^{n-1}} \int_{\partial B_\varepsilon(a)} u(x) \, dA$$

since the “exterior” normal field to $\Omega \setminus B_\varepsilon(a)$ points towards the midpoint $a$ along the boundary part $\partial B_\varepsilon(a)$.

By the continuity of $u$ at the point $a$ we obtain:

$$\frac{c}{\varepsilon^{n-1}} \int_{\partial B_\varepsilon(a)} u(x) \, dA \leq \max_{x \in \partial B_\varepsilon(a)} u(x) \frac{c}{\varepsilon^{n-1}} \int_{\partial B_\varepsilon(a)} dA = cnV_\varepsilon \max_{\partial B_\varepsilon(a)} u$$

but also

$$\frac{c}{\varepsilon^{n-1}} \int_{\partial B_\varepsilon(a)} u(x) \, dA \geq \min_{\partial B_\varepsilon(a)} u \frac{c}{\varepsilon^{n-1}} \int_{\partial B_\varepsilon(a)} dA = cnV_\varepsilon \min_{\partial B_\varepsilon(a)} u,$$

so that for $\varepsilon \to 0$ we have

$$- \int_{\partial B_\varepsilon(a)} u \frac{\partial v}{\partial n} \, dA \xrightarrow{\varepsilon \to 0} cnV_\varepsilon u(a).$$

Furthermore:

$$\left| \int_{\partial B_\varepsilon(a)} v \frac{\partial u}{\partial n} \, dA \right| = |\Psi_a(\varepsilon)\int_{\partial B_\varepsilon(a)} \frac{\partial u}{\partial n} \, dA| \leq \max_{\partial B_\varepsilon(a)} \left| \frac{\partial u}{\partial n} \right| \Psi_a(\varepsilon) \int_{\partial B_\varepsilon(a)} dA$$

$$= \max_{\partial B_\varepsilon(a)} \left| \frac{\partial u}{\partial n} \right|, \begin{cases} \varepsilon \ln \varepsilon, & n = 2 \\ \varepsilon, & n \geq 3 \end{cases} \xrightarrow{\varepsilon \to 0} 0.$$

For the left hand side of (3.5) we obtain

$$\int_{\Omega \setminus B_\varepsilon(a)} v \Delta u \, dx = \int_{\Omega} \chi_{\Omega \setminus B_\varepsilon(a)} v \Delta u \, dx$$

with the characteristic function $\chi_{\Omega \setminus B_\varepsilon(a)}(x) = \begin{cases} 1, & x \in \Omega \setminus B_\varepsilon(a) \\ 0, & \text{elsewhere}. \end{cases}$ Since $v$ is integrable (cf. Prop. 3.3 vi)) we can apply Lebesgue’s dominated convergence theorem to conclude

$$\int_{\Omega \setminus B_\varepsilon(a)} v \Delta u \, dx \xrightarrow{\varepsilon \to 0} \int_{\Omega} v \Delta u \, dx.$$
Thus, sending $\varepsilon \to 0$ in (3.5) we deduce
\[
\int_{\Omega} v \Delta u \, dx = \int_{\partial \Omega} \left( v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) \, dA + c \, nV_n u(a). \tag{3.6}
\]

Now we choose $c := \frac{1}{nV_n}$, so that

\[
\Psi_a(|x - a|) = \begin{cases} 
\frac{1}{2\pi} \ln |x - a|, & n = 2, \\
\frac{1}{n(2-n)V_n} |x - a|^{2-n}, & n \geq 3,
\end{cases} \tag{3.7}
\]

and (3.6) becomes

\[
u(a) = \int_{\Omega} \Psi_a(|x - a|) \Delta u \, dx + \int_{\partial \Omega} \left\{ u \frac{\partial (\Psi_a(|x - a|))}{\partial \nu} - \Psi_a(|x - a|) \frac{\partial u}{\partial \nu} \right\} \, dA. \tag{3.8}
\]

**Corollary 3.4 (Regularity of harmonic functions).** Let $u \in C^2(\Omega; \mathbb{R}) \cap C^1(\bar{\Omega}; \mathbb{R})$ be harmonic ($\Delta u \equiv 0$). Then $u \in C^\infty(\Omega; \mathbb{R})$.

**Proof.** By the representation formula (3.8) we have for all $a \in \Omega$:

\[
u(a) = \int_{\partial B(a)} u(x) \, dA(x) + \int_{B(a)} \{ \Psi_a(|x - a|) - \Psi_a(R) \} \Delta u \, dx
\]

Since $\Psi_a(|x - a|)$ is in $C^\infty$ in $x$ and $a$ for $x \neq a$ we can form derivatives of $u$ with respect to $a \in \Omega$ of all orders under the integral sign, meaning that $u \in C^\infty$.

**Proposition 3.5.** Let $u \in C^2(\Omega; \mathbb{R})$ and $B_R(a) \subset \subset \Omega$. Then

\[
u(a) = \int_{\partial B(a)} u(x) \, dA(x) + \int_{B(a)} \{ \Psi_a(|x - a|) - \Psi_a(R) \} \Delta u \, dx
\]

with the spherical mean value

\[
\int_{\partial B(a)} u(x) \, dA(x) := \frac{1}{nV_n R^{n-1}} \int_{\partial B(a)} u(x) \, dA(x) = \frac{1}{nV_n R^{n-1}} \int_{\partial B(a)} u \, dA.
\]

**Proof.** By the representation formula (3.8) with $\Omega = B_R(a)$ we have

\[
u(a) = \int_{B(a)} \Psi_a(|x - a|) \Delta u \, dx + \int_{\partial B(a)} \left\{ u \frac{\partial (\Psi_a(|x - a|))}{\partial \nu} - \Psi_a(|x - a|) \frac{\partial u}{\partial \nu} \right\} \, dA.
\]

For the last integral we obtain:

\[
- \int_{\partial B(a)} \left[ \Psi_a(|x - a|) \frac{\partial u}{\partial \nu} \right] dA = -\Psi_a(R) \int_{\partial B(a)} \frac{\partial u}{\partial \nu} \, dA \quad \text{div. thm Sec. 3.2}\quad -\Psi_a(R) \int_{B(a)} \Delta u \, dx = - \int_{B(a)} \Psi_a(R) \Delta u \, dx.
\]

Thus, rearranging the integrals we conclude

\[
u(a) = \int_{B(a)} \{ \Psi_a(|x - a|) - \Psi_a(R) \} \Delta u \, dx + \frac{1}{nV_n R^{n-1}} \int_{\partial B(a)} u \, dA.
\]

\[
= \int_{B(a)} \{ \Psi_a(|x - a|) - \Psi_a(R) \} \Delta u \, dx + \int_{\partial B(a)} u \, dA. \quad \square
\]
Corollary 3.6 (Mean value property of harmonic functions). Let $u \in C^2(\Omega; \mathbb{R})$ be harmonic. Then $u$ has the mean value property, i.e. it holds

$$u(a) = \frac{1}{\mathcal{L}^n(B_R(a))} \int_{B_R(a)} u(x) \, dx$$

(MVP)

for all $a \in \Omega$ and all $R > 0$ such that $B_R(a) \subset \subset \Omega$ (i.e. $0 < R < \text{dist}(a, \partial \Omega)$).

Definition. A continuous function $u \in C^0(\Omega; \mathbb{R})$ is said to satisfy/have the mean value property iff (MVP) is fulfilled for all $a \in \Omega$ and all $0 < R < \text{dist}(a, \partial \Omega)$.

Remark 3.7 (Solid mean value property). If $u \in C^0(\Omega; \mathbb{R})$ has the mean value property (MVP) then it also holds

$$u(a) = \frac{1}{n V_n r^{n-1}} \int_{\partial B_r(a)} u(x) \, dA = \frac{1}{n V_n R^n} \int_{B_R(a)} u(x) \, dx,$$

for all $a \in \Omega$ and all $0 < R < \text{dist}(a, \partial \Omega)$. The converse is also true.

Proof. We have

$$u(a) = \frac{1}{n V_n r^{n-1}} \int_{\partial B_r(a)} u(x) \, dA = \frac{1}{n V_n R^n} \int_{B_R(a)} u(x) \, dA,$$

An integration with respect to $r$ from 0 to $R$ gives

$$n V_n u(a) \int_0^R r^{n-1} \, dr = \int_0^R \int_{\partial B_r(a)} u(x) \, dA \, dr \overset{\text{Cavalieri}}{=} \int_{B_R(a)} u(x) \, dx$$

so that we obtain $u(a) = \int_{B_R(a)} u(x) \, dx$. $\square$

Theorem 3.8 (Koebe). If $u \in C^0(\Omega; \mathbb{R})$ has the mean value property (MVP), i.e. it holds

$$u(a) = \int_{\partial B_R(a)} u(x) \, dA$$

for all $a \in \Omega$ and all $0 < R < \text{dist}(a, \partial \Omega)$.

Then $u \in C^\infty(\Omega; \mathbb{R})$ and $\Delta u \equiv 0$ in $\Omega$.

Proof. Note that $u$ is assumed to be only continuous, meaning that we cannot use the preceding results were we had at least $C^2$. In order to conclude the regularity of $u$ we consider the mollifier for arbitrary $\varepsilon > 0$:

$$k_\varepsilon(s) := \begin{cases} \exp \left( \frac{1}{s^2 - \varepsilon^2} \right), & \text{for } 0 \leq |s| \leq \varepsilon, \\ 0, & \text{elsewhere.} \end{cases}$$

From the Analysis II course we know that $k_\varepsilon \in C^\infty(\mathbb{R}; \mathbb{R})$ with $\text{supp} \, k_\varepsilon = [-\varepsilon, \varepsilon]$.

The mean value property of $u$ gives us

$$n V_n r^{n-1} u(a) = \int_{\partial B_r(a)} u(x) \, dA \quad \forall 0 < r < \text{dist}(a, \partial \Omega)$$

and by a multiplication with $k_\varepsilon(r)$ we obtain

$$n V_n r^{n-1} k_\varepsilon(r) u(a) = k_\varepsilon(r) \int_{\partial B_r(a)} u(x) \, dA.$$
Let $0 < \varepsilon < \text{dist}(a, \partial \Omega)$, and integrate with respect to $r$ from 0 to $\varepsilon$:

$$nV_n u(a) \int_0^\varepsilon r^{n-1} k_\varepsilon(r) \, dr = \int_0^\varepsilon k_\varepsilon(r) \int_{\partial B_r(a)} u \, dA \, dr = \int_0^\varepsilon \int_{\partial B_r(a)} k_\varepsilon(r) u \, dA \, dr \overset{\text{Cavalieri}}{=} \int_{B_a(a)} k_\varepsilon(|x-a|) u(x) \, dx.$$ Thus,

$$u(a) = \frac{1}{nV_n} \left( \int_0^\varepsilon r^{n-1} k_\varepsilon(r) \, dr \right)^{-1} \int_{B_a(a)} k_\varepsilon(|x-a|) u(x) \, dx \geq c(n, \Omega) \int_{B_\varepsilon(a)} k_\varepsilon(|x-a|) u(x) \, dx.$$ Since $k_\varepsilon(|x-a|)$ is in $C^\infty$ in $x$ and $a$ and vanishes outside the ball $B_\varepsilon(a)$ we can form derivatives of $u$ with respect to $a \in \Omega$ of all orders under the integral sign, meaning that $u \in C^\infty$.

Now, Proposition 3.5 is applicable to conclude the harmonicity of $u$: For all $B_R(a) \subset \subset \Omega$ we have:

$$u(a) = \int_{\partial B_R(a)} u \, dA + \iint_{B_R(a)} \{ \Psi_a(|x-a|) - \Psi_a(R) \} \Delta u \, dx \Rightarrow \iint_{B_R(a)} \{ \Psi_a(|x-a|) - \Psi_a(R) \} \Delta u \, dx = 0.$$ Since $\Psi_a$ is strictly increasing (see Prop. 3.3 ii) and $c = \frac{1}{nV_n} > 0$ we have

$$\Psi_a(r) - \Psi_a(R) < 0 \quad \forall \ 0 < r < R \quad \Rightarrow \quad \Delta u \equiv 0,$$

since otherwise assuming $\Delta u(a) \neq 0$, e.g. $\Delta u(a) > 0 \overset{u \in C^\infty}{\Rightarrow} \Delta u > 0$ in some $B_\varepsilon(a) \Rightarrow \iint_{B_\varepsilon(a)} \ldots < 0 \quad \square.$

**Theorem 3.9** (Harnack inequality). Let $u \in C^2(\Omega; \mathbb{R}_{\geq 0})$ be harmonic and non-negative ($u \geq 0$ in $\Omega$). Then for all $R > 0$ such that $B_4R(a) \subset \subset \Omega$ it holds

$$\sup_{B_R(a)} u \leq 3^n \inf_{\varepsilon B_R(a)} u.$$

**Remark 3.10.** Note that the constant on the right hand side only depends on the dimension and *not* on the domains or function!

**Proof of Theorem 3.9.** Let $x, y \in B_R(a)$ be arbitrary chosen. By the mean value property of $u$ on $B_R(a)$ we obtain

$$u(x) = \frac{1}{V_n R^n} \int_{B_R(x)} u(\xi) \, d\xi \overset{u \geq 0}{\leq} \frac{1}{V_n R^n} \int_{B_{2R}(a)} u(\xi) \, d\xi,$$

and by the mean value property of $u$ on $B_3R(y) \subset B_{4R}(a) \subset \subset \Omega$:

$$u(y) = \frac{1}{V_n (3R)^n} \int_{B_{3R}(y)} u(\xi) \, d\xi \overset{u \geq 0}{\geq} \frac{1}{3^n V_n R^n} \int_{B_{2R}(a)} u(\xi) \, d\xi \geq \frac{1}{3^n} u(x).$$

Thus, it holds: $u(x) \leq 3^n u(y)$ for all $x, y \in B_R(a)$. Take the supremum over all $x$ gives us: $\sup_{B_R(a)} u \leq 3^n \inf_{\varepsilon B_R(a)} u$. $\square$
Corollary 3.11 (Liouville type). Let \( u \in C^2(\mathbb{R}^n; \mathbb{R}) \) be harmonic (on the entire \( \mathbb{R}^n \)) with \( u \geq c > -\infty \) on \( \mathbb{R}^n \) (or \( u \leq c < +\infty \)). Then \( u \equiv \text{const.} \)

Proof. Let \( u \geq c > -\infty \) on \( \mathbb{R}^n \), then \( \inf_{\mathbb{R}^n} u \in \mathbb{R} \) and for the function \( v := u - \inf_{\mathbb{R}^n} u \) it holds:

\[
\Delta v = \Delta u \equiv 0, \quad v \geq 0 \quad \text{on} \quad \mathbb{R}^n.
\]

Hence, we can apply Harnack’s inequality and obtain for all \( R > 0 \):

\[
\sup_{B_R(0)} v \leq 3^n \inf_{B_R(0)} v. \tag{3.9}
\]

We have

\[
\inf_{B_R(0)} v = \inf_{B_R(0)} u - \inf_{\mathbb{R}^n} u \xrightarrow{R \to \infty} 0.
\]

Thus, sending \( R \to \infty \) in (3.9) we obtain

\[
\sup_{\mathbb{R}^n} v = 0,
\]

and since \( v \geq 0 \) we conclude \( v \equiv 0 \) and so \( u \equiv \inf_{\mathbb{R}^n} u = \text{const.} \)

Proposition 3.12 (Harnack inequality on arbitrary domains). Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, \( u \in C^2(\Omega; \mathbb{R} \geq 0) \) be harmonic and non-negative. Then for any \( \Omega' \subset \subset \Omega \) there exists a constant \( c = c(\Omega', \Omega) > 0 \) such that

\[
\sup_{\Omega'} u \leq c \inf_{\Omega'} u.
\]

Proof. (Idea: Harnack inequality on balls & covering argument)

Since \( \Omega' \subset \subset \Omega \) we have

\[
\text{dist}(\Omega', \partial \Omega) = \inf_{x \in \Omega', y \in \partial \Omega} |x - y| > 0.
\]

Thus, we can choose an \( R > 0 \) with \( 4R < \text{dist}(\Omega', \partial \Omega) \). Cover \( \overline{\Omega'} \) by \( N \) open balls of radius \( R \) (possible due to Heine-Borel, since \( \overline{\Omega'} \) is compact). Note also that \( N = N(\Omega', \Omega) \).

Let \( x, y \in \overline{\Omega'} \) be arbitrary. Then there exist points \( x_2, x_3, \ldots, x_{p-1} \), such that with \( x_1 = x, x_p = y, p \leq N \) it holds

\[
x_j \text{ and } x_{j+1} \text{ belong to one of these balls.}
\]

Now we can apply Theorem 3.9 (the Harnack inequality on balls):

\[
u(x) = u(x_1) \leq 3^n u(x_2) \leq 3^{2n} u(x_3) \leq \ldots \leq 3^{pn} u(x_p) = 3^{pn} u(y) \leq 3^n \inf_{\Omega'} u(y) \quad \forall x, y \in \overline{\Omega'}.
\]

Hence,

\[
\sup_{\Omega'} u \leq 3^n \inf_{\Omega'} u,
\]

whereby \( N = N(\Omega', \Omega) \) denotes the number of balls which is needed to cover \( \overline{\Omega'} \) by open balls of radius \( R < \frac{1}{4} \text{dist}(\Omega', \partial \Omega) \).
3.4 Convergence theorems

**Theorem 3.13** (Weierstraß-Harnack). The limit of a locally uniformly convergent sequence of harmonic functions \( \{u_k\}_{k \in \mathbb{N}} \) is harmonic.

**Remark 3.14.** 1) “locally uniformly” means uniform convergence on any compact subset.

2) Recall the implications:

\[
\text{uniform convergence} \quad \Rightarrow \quad \text{local uniform convergence} \quad \Rightarrow \quad \text{pointwise convergence}
\]

**Proof of Theorem 3.13.** Let us denote by \( u \) the limit functions (it exists pointwisely). By the uniform convergence of the continuous functions \( u_k \) we conclude that also \( u \) is continuous. Let now \( B_R(a) \subset \subset \Omega \), by the “solid” (mvp) (see Rem. 3.7) of \( u_k \) we obtain:

\[
u(a) \xrightarrow{k \to \infty} u_k(a) = \int_{B_R(a)} u_k(x) \, dx \quad \text{uniformly} \quad \rightarrow \quad \int_{B_R(a)} u(x) \, dx,
\]

meaning that \( u \) has the mean value property, so that by Thm 3.8 we obtain the harmonicity of \( u \).

**Theorem 3.15** (Harnack’s convergence theorem). Let \( \{u_k\}_{k \in \mathbb{N}} \) be a monotone increasing (or decreasing) sequence of harmonic functions in a domain \( \Omega \), and suppose that for some point \( a \in \Omega \), the sequence \( \{u_k(a)\}_{k \in \mathbb{N}} \) is bounded. Then the sequence \( \{u_k\}_{k \in \mathbb{N}} \) converges uniformly on any \( \Omega' \subset \subset \Omega \) to a harmonic function.

**Proof.** Consider an arbitrary \( \Omega' \subset \subset \Omega \) w.l.o.g. \( a \in \Omega' \).

Since \( \{u_k(a)\}_{k \in \mathbb{N}} \) is bounded and increasing it is convergent, thus, for the increasing sequence we obtain

\[
0 \leq u_m(a) - u_l(a) < \varepsilon \quad \forall m \geq l > N = N(\varepsilon).
\]

Hence,

\[
\sup_{x \in \Omega'} |u_m(x) - u_l(x)| = \sup_{x \in \Omega'} (u_m(x) - u_l(x)) \overset{\text{Harnack on domains Prop. 3.12}}{\leq} c(\Omega', \Omega) \inf_{x \in \Omega'} (u_m(x) - u_l(x)) \leq c(\Omega', \Omega)(u_m(a) - u_l(a))
\]

\[
\leq c(\Omega', \Omega) \varepsilon \quad \forall m \geq l > N = N(\varepsilon),
\]

i.e. \( \{u_k\}_{k \in \mathbb{N}} \) converges uniformly on \( \Omega' \) and by Theorem 3.13 the limit function is harmonic.

\[
\Box
\]

3.5 Interior estimates of derivatives

**Theorem 3.16** (A priori estimates for harmonic functions). Let \( u \in C^2(\Omega; \mathbb{R}) \) be harmonic and \( B_R(a) \subset \subset \Omega \). Then,

\[
|D^\alpha u(a)| \leq c(n, \alpha) R^{-|\alpha|} \sup_{B_R(a)} |u| \quad \forall \alpha \in \mathbb{N}_0^n.
\]

**Proof.** Idea: Use induction over the length of the multi-index.

Since \( u \) is harmonic we have \( u \in C^\infty \) and all \( D^\alpha u \) are also harmonic for all \( \alpha \in \mathbb{N}_0^n \), indeed, by Schwarz theorem:

\[
\Delta D^\alpha u = D^\alpha \Delta u \equiv 0.
\]

For \( |\alpha| = 0 \) we have trivially by the definition of the supremum: \( u(a) \leq \sup_{B_R(a)} |u| \).

For \( |\alpha| = \alpha_1 + \ldots + \alpha_n = 1 \) we have \( D^\alpha u = \partial_j u \) for some \( j \in \{1, \ldots, n\} \), and since \( \partial_j u \) is harmonic an application of the "solid" (mvp) (see Rem. 3.7) gives us

\[
\partial_j u(a) = \frac{1}{V_n R^n} \int_{B_R(a)} \partial_j u \, dx \overset{\text{fund. thm calc.}}{=} \frac{1}{V_n R^n} \int_{\partial B_R(a)} \nu_j u \, dA,
\]

and since \( |\nu| = 1 \):

\[
|\partial_j u(a)| \leq \frac{1}{V_n R^n} \sup_{\partial B_R(a)} |u| \int_{\partial B_R(a)} dA \leq \frac{n}{R} \sup_{B_R(a)} |u|.
\]

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Let now \( \xi \in B_{R/2}(a) \) be arbitrary, then using the same argument as above we obtain

\[
|\partial_i u(\xi)| \leq \frac{1}{V_n(R/2)^n} \left| \int_{B_{R/2}(\xi)} \partial_i u \, dx \right| \leq \frac{2n}{R} \sup_{B_{R/2}(\xi)} |u| | \leq \frac{2n}{R} \sup_{B_R(a)} |u|.
\]

Then, for all \( \xi \in B_{R/4}(\xi) \) it holds

\[
|\partial_i \partial_j u(\xi)| \leq \frac{2n}{R^2} \sup_{B_{R/2}(a)} |\partial_i u| \leq \frac{8n^2}{R^2} \sup_{B_R(a)} |u|.
\]

The conclusion follows by an induction argument: Assume that for \( |\alpha| = k \) it holds

\[
|D^\alpha u(\xi)| \leq \frac{c(n, \alpha)}{R^{|\alpha|}} \sup_{B_R(a)} |u| \quad \forall \xi \in B_{R/2^k}(a).
\]

Then for all \( \xi \in B_{R/2^{k+1}}(a) \) we conclude:

\[
|\partial_j D^\alpha u(\xi)| \leq \frac{c(n)}{R} \sup_{B_{R/2^k}(a)} |D^\alpha u| \leq \frac{c(n, \alpha)}{R^{|\alpha|+1}} \sup_{B_R(a)} |u|,
\]

and especially the estimate holds true for \( \xi = a \).

3.6 Maximum principles

**Definition.** Let \( \Omega \subseteq \mathbb{R}^n \) be open. A function \( u \in C^0(\Omega; \mathbb{R}) \) is called subharmonic iff

\[
u(a) = \inf_{\partial B_R(a)} u(x) dA(x) \quad \forall a \in \Omega, \quad \forall R > 0, \text{ s.t. } B_R(a) \subset \subset \Omega.
\]

A function \( u \in C^0(\Omega; \mathbb{R}) \) is called superharmonic iff

\[
u(a) = \sup_{\partial B_R(a)} u(x) dA(x) \quad \forall a \in \Omega, \quad \forall R > 0, \text{ s.t. } B_R(a) \subset \subset \Omega.
\]

**Corollary 3.17.** Let \( u \in C^2(\Omega; \mathbb{R}) \). Then

a) \( u \) is subharmonic iff \( \Delta u(x) \geq 0 \) for all \( x \in \Omega \).

b) \( u \) is superharmonic iff \( \Delta u(x) \leq 0 \) for all \( x \in \Omega \).

**Proof.** We start to prove a) and assume \( \Delta u \geq 0 \). Then by the representation formula from Proposition 3.5 we have for any \( B_R(a) \subset \subset \Omega \):

\[
u(a) = \int_{\partial B_R(a)} u \, dA + \int_{B_R(a)} \left\{ \Psi_a(|x-a|) - \Psi_a(R) \right\} \Delta u \, dx \leq \int_{\partial B_R(a)} u \, dA,
\]

meaning that \( u \) is subharmonic. If in the contrary \( u \) is assumed to be subharmonic and there exists a point \( \xi \in \Omega \) such that \( \Delta u(\xi) < 0 \), then by continuity we have \( \Delta u < 0 \) in a small ball \( B_r(\xi) \) and from Proposition 3.5 we obtain

\[
u(a) = \int_{\partial B_r(\xi)} u \, dA + \int_{B_r(\xi)} \left\{ \Psi_a(|x-a|) - \Psi_a(R) \right\} \Delta u \, dx > \int_{\partial B_r(a)} u \, dA,
\]

in contrast to the subharmonicity of \( u \).

Similar arguments prove part b).
Corollary 3.18. a) Let \( u \in C^0(\Omega; \mathbb{R}) \) be subharmonic. Then
\[
u(a) \leq \int_{B_R(a)} u(x) \, dx \quad \forall a \in \Omega, \forall R > 0, \text{ s.t. } B_R(a) \subset \subset \Omega.
\]
b) Let \( u \in C^0(\Omega; \mathbb{R}) \) be superharmonic. Then
\[
u(a) \geq \int_{B_R(a)} u(x) \, dx \quad \forall a \in \Omega, \forall R > 0, \text{ s.t. } B_R(a) \subset \subset \Omega.
\]

Theorem 3.19 (Strong maximum principle for subharmonic functions). Let \( \Omega \subseteq \mathbb{R}^n \) be a domain and \( u \in C^0(\Omega; \mathbb{R}) \) be subharmonic. If there exists a point \( a \in \Omega \) such that
\[
u(a) = \sup_{\Omega} u,
\]
then \( u \equiv \text{const} \) in \( \Omega \).

Proof. Since by the assumptions of the theorem the supremum is attained there exists an \( M \in \mathbb{R} \) such that \( \sup_{\Omega} u = M \). Moreover,
\[
a \in \Omega_M := \{x \in \Omega : u(x) = M\} \neq \emptyset.
\]
Since \( u \) is continuous, \( \Omega_M \) is relatively closed in \( \Omega \). Let \( z \in \Omega_M \) and \( B_R(z) \subset \subset \Omega \), then
\[
M = u(z) \leq \int_{B_R(a)} u(x) \, dx \leq M,
\]
so that equality holds and we obtain \( u \equiv M \) in \( B_R(z) \), i.e. \( B_R(z) \subseteq \Omega_M \) meaning that \( \Omega_M \) (open) \( \subseteq \Omega \). Thus, \( \Omega_M \) is both open and relatively closed in \( \Omega \), and since \( \Omega \) is connected we conclude \( \Omega_M = \Omega \).

Theorem 3.20 (Strong minimum principle for superharmonic functions). Let \( \Omega \subseteq \mathbb{R}^n \) be a domain and \( u \in C^0(\Omega; \mathbb{R}) \) be superharmonic. If there exists a point \( a \in \Omega \) such that
\[
u(a) = \inf_{\Omega} u,
\]
then \( u \equiv \text{const} \) in \( \Omega \).

Proof. Instead of \( u \) consider \( -u \) and apply Theorem 3.19.

Corollary 3.21. Let \( \Omega \subseteq \mathbb{R}^n \) be a domain and \( u \in C^0(\Omega; \mathbb{R}) \) have the mean value property (MVP), i.e. \( u \) is harmonic (cf. Thm. 3.8). Then \( u \) cannot attain an interior maximum or minimum unless \( u \) is constant.

Corollary 3.22 (Weak maximum principle for subharmonic functions). Let \( \Omega \subseteq \mathbb{R}^n \) be open and bounded and \( u \in C^0(\Omega; \mathbb{R}) \) be subharmonic. Then
\[
\max_{\Omega} u = \max_{\partial \Omega} u.
\]

Proof. Follows from strong maximum principle applied on a connected component (Ger.: Zusammenhangskomponente).

Corollary 3.23 (Weak minimum principle for superharmonic functions). Let \( \Omega \subseteq \mathbb{R}^n \) be open and bounded and \( u \in C^0(\Omega; \mathbb{R}) \) be superharmonic. Then
\[
\min_{\Omega} u = \min_{\partial \Omega} u.
\]

Corollary 3.24. Let \( \Omega \subseteq \mathbb{R}^n \) be open and bounded and \( u \in C^0(\Omega; \mathbb{R}) \) have the mean value property (MVP), i.e. \( u \) is harmonic (cf. Thm. 3.8). Then
\[
\min_{\partial \Omega} u \leq u(x) \leq \max_{\partial \Omega} u \quad \forall x \in \Omega.
\]

\(^2\text{meaning in the subspace topology, i.e. there exists a closed set } A \subseteq \mathbb{R}^n \text{ such that } \Omega_M = \Omega \cap A, \text{ e.g. } (0, \frac{1}{2}) \text{(relatively closed)} \subset (0, 1). \)
Consider the function $w := u - v$. Then $w$ is harmonic in $\Omega$ and vanishes on the boundary $\partial \Omega$, so that by Corollary 3.24 we conclude $w \equiv 0$ in $\Omega$.

**Proof.** Consider the function $w := u - v$. Then $w$ is harmonic in $\Omega$ and vanishes on the boundary $\partial \Omega$, so that by Corollary 3.24 we conclude $w \equiv 0$ in $\Omega$.

**Corollary 3.26** (Term “subharmonic”). Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $u, v \in C^0(\partial \Omega; \mathbb{R})$ be such that $u$ is harmonic and $v$ is superharmonic with $u = v$ on $\partial \Omega$. Then for all $x \in \Omega$ we have $v(x) \leq u(x)$.

**Proof.** Consider $w := v - u$. Then $w$ is subharmonic in $\Omega$ and $w \equiv 0$ on $\partial \Omega$, so that by Corollary 3.22 we obtain $\max_{\Omega} w = \max_{\partial \Omega} w = 0$, hence,

$$v(x) - u(x) \leq \max_{\partial \Omega} (v - u) = 0.$$

**Corollary 3.27** (Term “superharmonic”). Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $u, v \in C^0(\partial \Omega; \mathbb{R})$ be such that $u$ is harmonic and $v$ is superharmonic with $u = v$ on $\partial \Omega$. Then for all $x \in \Omega$ we have $v(x) \geq u(x)$.

### 3.7 Green’s functions

In the following we denote by $\Omega \subset \mathbb{R}^n$ a bounded Lipschitz domain. Let $w \in C^2(\Omega; \mathbb{R}) \cap C^1(\overline{\Omega}; \mathbb{R})$ be harmonic. Then

$$G(x, a) := \Psi_a(|x - a|) + w(x)$$

is again a fundamental solution of the Laplace equation with singularity in $a$ and (3.8) stays valid when we replace $\Psi_a$ by $G$:

$$u(a) = \int_{\Omega} G(x, a) \Delta u \, dx + \int_{\partial \Omega} \left\{ u \frac{\partial G(x, a)}{\partial n} - G(x, a) \frac{\partial u(x)}{\partial n} \right\} \, dA(x)$$

(just add (3.8) and Green’s second identity with $u$ and $w$). If, in particular, $G(x, a) = 0$ for all $x \in \partial \Omega$ (and $a \in \Omega$), then $G$ is called the Green’s function for the domain $\Omega$.

**Corollary 3.28.** Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. If $u \in C^2(\Omega; \mathbb{R}) \cap C^1(\overline{\Omega}; \mathbb{R})$ solves the Dirichlet boundary value problem

$$\begin{cases}
\Delta u = f & \text{in } \Omega, \\
u \in \varphi & \text{on } \partial \Omega,
\end{cases}$$

then $u$ has the representation

$$u(a) = \int_{\Omega} G(x, a) f(x) \, dx + \int_{\partial \Omega} \varphi(x) \frac{\partial G(x, a)}{\partial n} \, dA(x) \quad \forall \ a \in \Omega. \quad (3.10)$$

**Remark 3.29.** The existence of the solution to the Dirichlet boundary value problem in not guaranteed by this formula! Also the existence of the Green’s function is not guaranteed! The construction of a Green’s function is in general a difficult matter, and can only be done when $\Omega$ has a simple geometry, e.g. is a half-space or a ball.

**Theorem 3.30** (Green’s function of the unit ball). The Green’s function of the unit ball $B_1(0)$ is given by

$$G(x, a) = \Psi_a \left( \sqrt{|x|^2 + |a|^2 - 2 \langle x, a \rangle} \right) - \Psi_a \left( \sqrt{|x|^2 |a|^2 - 2 \langle x, a \rangle + 1} \right) \quad \forall \ x \in B_1(0), \forall \ a \in B_1(0), \ x \neq a.$$

**Remark 3.31.** We have $G(x, a) = G(a, x) \quad \forall \ x \neq a$. 

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Proof of Theorem 3.30. Let \( a \in B_1(0) \). We have to show that
\[
  w(x) := \Psi_a \left( \sqrt{|x|^2 |a|^2 - 2 \langle x, a \rangle + 1} \right)
\]
belongs to \( C^2(B_1(0); \mathbb{R}) \cap C^1(\overline{B_1(0)}; \mathbb{R}) \) and is harmonic. Note that
\[
  \sqrt{|x|^2 |a|^2 - 2 \langle x, a \rangle + 1} = \begin{cases} 
    |a| \frac{|a|^2 - x}{|a|^2} & , \quad a \neq 0, \\
    1, & , \quad a = 0,
  \end{cases}
\]
and the argument vanishes iff \( x = \frac{a}{|a|^2} \Rightarrow |x| = \frac{1}{|a|} > 1 \). Hence, \( w \in C^\infty(\overline{B_1(0)}; \mathbb{R}) \) and we obtain
\[
  w(x) = \begin{cases} 
    \frac{1}{n \pi^2} \ln \left( |x|^2 |a|^2 - 2 \langle x, a \rangle + 1 \right)^{1/2}, & , \quad n = 2 \\
    \frac{1}{n(2-n) \pi^2} \left( |x|^2 |a|^2 - 2 \langle x, a \rangle + 1 \right)^{\frac{2-n}{2}}, & , \quad n \geq 3.
  \end{cases}
\]
Thus, for all \( n \geq 2 \)
\[
  \nabla_x (\nabla_x w)(x) = \frac{1}{n \pi^2} \left( |x|^2 |a|^2 - 2 \langle x, a \rangle + 1 \right)^{-n/2} \langle |a|^2 x - a \rangle
\]
\[
  D_x \nabla_x w(x) = - \frac{\left( |x|^2 |a|^2 - 2 \langle x, a \rangle + 1 \right)^{\frac{2-n}{2}}}{V_n} \left\{ |a|^4 x \otimes x - |a|^2 a \otimes x - |a|^2 x \otimes a + a \otimes a \right. 
\]
\[
  - \left. \left( |x|^2 |a|^2 - 2 \langle x, a \rangle + 1 \right) \frac{|a|^2}{n} E_n \right\}
\]
\[
  \Rightarrow \Delta_x w(x) = 0 \quad \text{in } B_1(0).
\]
Furthermore for \( x \in \partial B_1(0) \): we have \( |x| = 1 \) and thus \( G(x, a) = 0 \). \( \Box \)

Corollary 3.32 (Poisson’s formula for harmonic functions in the unit ball). If \( u \in C^2(B_1(0); \mathbb{R}) \cap C^1(\overline{B_1(0)}; \mathbb{R}) \) is harmonic in \( B_1(0) \) and satisfies \( u = \varphi \) on \( \partial B_1(0) \), then:
\[
  u(a) = \frac{1 - |a|^2}{n V_n} \int_{\partial B(0)} \frac{\varphi(x)}{|x - a|^n} \, dA(x) \quad \forall \, a \in B_1(0).
\]

Remark 3.33. This is not yet the solution to the Dirichlet boundary value problem on the unit ball, but we are already approaching it.

Proof of Corollary 3.32. By the representation formula (3.10) we only need to compute the normal derivative of \( G \) along \( \partial B_1(0) \): we have
\[
  \frac{\partial G(x, a)}{\partial \nu(x)} = \langle \nabla_x \Psi_a, \nu(x) \rangle - \langle \nabla_x w(x), \nu(x) \rangle
\]
\[
  \text{Prop. 3.3 iii)} \left\{ \frac{x - a}{|x - a|^n}, \nu(x) \right\} - \frac{1}{n V_n} \left( |x|^2 |a|^2 - 2 \langle x, a \rangle + 1 \right)^{-n/2} \langle |a|^2 x - a, \nu(x) \rangle
\]
\[
  \text{Prop. 3.3 iii)} \left\{ \frac{x - a}{|x - a|^n}, \nu(x) \right\} - \frac{1}{n V_n} \frac{1}{|x - a|^{n-2}} \left( |x|^2 |a|^2 - 2 \langle x, a \rangle + 1 \right) \langle |a|^2 x - a, \nu(x) \rangle
\]
\[
  = \frac{|x|^2 - \langle a, x \rangle - |a|^2 |x|^2 + \langle a, x \rangle |x|^2}{n V_n |x - a|^n} \quad \text{Prop. 3.3 iii)}.
\]
\( \Box \)

Remark 3.34. Use an approximation argument to show that Poisson’s formula continues to hold for \( u \in C^2(B_1(0); \mathbb{R}) \cap C^0(\overline{B_1(0)}; \mathbb{R}) \).
Theorem 3.35 (Existence of solutions to the Dirichlet boundary value problem on the unit ball). Let \( \varphi \in C^0(\partial B_1(0); \mathbb{R}) \). Then

\[
u(a) := \begin{cases} 
1 - |a|^2/nV_n & \int_{\partial B_1(0)} \frac{\varphi(x)}{|x-a|^{n+2}} \, dA(x), \quad a \in B_1(0), \\
\varphi(a), & a \in \partial B_1(0)
\end{cases}
\]

belongs to \( C^2(B_1(0); \mathbb{R}) \cap C^0(\overline{B_1(0)}; \mathbb{R}) \) and solves the boundary value problem

\[
\begin{align*}
\Delta u &\equiv 0, \quad \text{in } B_1(0), \\
u &\equiv \varphi, \quad \text{on } \partial B_1(0).
\end{align*}
\]

Proof. First note that differentiation with respect to the variable \( a \) is allowed due to smoothness and compactness reasons and we obtain

\[
\Delta^a u(a) = \Delta^a \int_{\partial B_1(0)} \varphi(x) \frac{\partial G(x,a)}{\partial \nu(x)} \, dA(x) = \int_{\partial B_1(0)} \varphi(x) \Delta^a \frac{\partial G(x,a)}{\partial \nu(x)} \, dA(x)
\]

\[
= \int_{\partial B_1(0)} \varphi(x) \frac{\partial}{\partial \nu(x)} \Delta^a G(x,a) \bigg|_{\partial B_1(0)} dA(x) = 0 \quad \forall \ a \in B_1(0),
\]

since \( G \) was our Green’s function. Thus, we only need to establish the continuity of \( u \) up to the boundary \( \partial B_1(0) \), i.e.

\[
\lim_{a \to b} \ a \in B_1(0) \quad u(a) = \varphi(b) \quad \forall \ b \in \partial B_1(0).
\]

Since \( \varphi \in C^0(\partial B_1(0); \mathbb{R}) \) we have: \( \forall \ \varepsilon > 0 \exists \ \delta > 0:

\[
|\varphi(x) - \varphi(b)| < \frac{\varepsilon}{2} \quad \forall \ x \in \partial B_1(0) \text{ s.t. } |x - b| < \delta.
\]

Applying Poisson’s formula (Cor. 3.32) to the harmonic function \( u \equiv 1 \) we obtain directly\(^3\)

\[
1 = \frac{1 - |a|^2}{nV_n} \int_{\partial B_1(0)} \frac{1}{|x-a|^{n}} dA(x) \quad \forall \ a \in B_1(0).
\]

Hence, for all \( a \in B_1(0) \) such that \( |a - b| < \frac{\delta}{2} \) we have

\[
|u(a) - \varphi(b)| \leq \frac{1}{nV_n} \int_{\partial B_1(0)} \frac{\|x-a\|^2}{|x-a|^{1+n}} dA(x) - \varphi(b) \int_{\partial B_1(0)} \frac{1}{|x-a|^{n}} dA(x) \leq \frac{1}{nV_n} \int_{\partial B_1(0)} \frac{|\varphi(x) - \varphi(b)|}{|x-a|^n} dA(x)
\]

\[
\leq \frac{1 - |a|^2}{nV_n} \left\{ \int_{\{x \in \partial B_1(0): |x - b| < \delta\}} \frac{|\varphi(x) - \varphi(b)|}{|x - a|^{n+2}} \, dA(x) + \int_{\{x \in \partial B_1(0): |x - b| \geq \delta\}} \frac{|\varphi(x) - \varphi(b)|}{|x - a|^{n+2}} \, dA(x) \right\}
\]

\[
\leq \frac{1 - |a|^2}{2nV_n} \int_{\partial B_1(0)} \frac{1}{|x-a|^{n+2}} \, dA(x) + \frac{1 - |a|^2}{2V_n} \max_{\partial B_1(0)} |\varphi| \int_{\{x \in \partial B_1(0): |x - b| \geq \delta\}} \frac{1}{|x - a|^{n+2}} \, dA(x)
\]

\[
\leq \frac{2(1 - |a|^2)}{2nV_n} \max_{\partial B_1(0)} |\varphi| \int_{\{x \in \partial B_1(0): |x - b| \geq \delta\}} \frac{1}{|x - a|^{n+2}} \, dA(x)
\]

\[
\leq \frac{2(1 - |a|^2)}{(\delta/2)^n} \leq \frac{\varepsilon}{2} + \frac{2(1 - |a|^2)}{(\delta/2)^n} < \varepsilon,
\]

whereby \( \delta \) was chosen in appropriate way, note also that \( 1 - |a|^2 \leq 2(1 - |a|) \leq 2 |b - a| < \delta \) since \( b \in \partial B_1(0) \) and \( |a - b| < \frac{\delta}{2} \).

\(^3\)good luck to evaluate the right hand side directly.\[\]

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To obtain Poisson’s integral formula on an arbitrary ball $B_R(\xi)$ with $\varphi \in C^0(\partial B_R(\xi); \mathbb{R})$ consider

$$\varphi_R(a) := \varphi(Ra + \xi) \quad \forall \ a \in \partial B_1(0).$$

Then $\varphi_R \in C^0(\partial B_1(0); \mathbb{R})$ and

$$u_R(a) := \frac{1 - |a|^2}{nV_n} \int_{\partial B_1(0)} \frac{\varphi_R(x)}{|x - a|^n} \, dA(x)$$

is the unique solution to the Dirichlet boundary value problem on the unit ball of the Laplace equation with boundary values prescribed by $\varphi$. For the rescaled function

$$u(a) := u_R \left( \frac{a - \xi}{R} \right) \quad \text{for } a \in B_R(\xi)$$

we obtain $u \in C^2(B_R(\xi); \mathbb{R}) \cap C^0(B_R(\xi); \mathbb{R})$, for all $a \in B_R(\xi)$ \quad $\Delta_a u(a) = \frac{1}{R^2} (\Delta_a u_R) \left( \frac{a - \xi}{R} \right) = 0$ and for all $b \in \partial B_R(\xi)$

$$u(b) = u_R \left( \frac{b - \xi}{R} \right) = \varphi_R \left( \frac{b - \xi}{R} \right) = \varphi(b),$$

i.e. this $u$ solves the Dirichlet problem on $B_R(\xi)$. Furthermore,

$$u(a) = u_R \left( \frac{a - \xi}{R} \right) = \frac{1 - |a - \xi|^2}{nV_n} \int_{\partial B_1(0)} \frac{\varphi_R(x)}{|x - a - \xi|^n} \, dA(x) = \frac{R^2 - |a - \xi|^2}{nV_n R^2} \int_{\partial B_1(0)} \frac{\varphi(x)}{R^2 - a - \xi| - R^2} \, dA(x)$$

Summarizing, we have the existence of solutions to the Dirichlet boundary value problem for continuous boundary values on any ball. Now we want to address the question of existence of solutions on arbitrary bounded domains.

### 3.8 The Dirichlet Problem: Perron’s method of subharmonic functions

This strategy goes back to the following paper by Oskar Perron: *Eine neue Behandlung der ersten Randwertaufgabe für $\Delta u = 0$*. Mathematische Zeitschrift 18 (1923), pp. 42–54.

**Definition** (Harmonic lifting). Let $u \in C^0(\Omega; \mathbb{R})$ and $B_R(\xi) \subset \subset \Omega$. The harmonic lifting of $u$ in $b_R(\xi)$ is given by

$$P_B u(a) := \begin{cases} u(a), & a \in \Omega \setminus B_R(\xi), \\ \frac{R^2 - |a - \xi|^2}{nV_n R^2} \int_{\partial B_R(\xi)} \frac{\varphi(y)}{|y - a|^n} \, dA(y), & a \in B_R(\xi). \end{cases}$$

Hence, $P_B u$ is harmonic in $B_R(\xi)$, coincides with $u$ in $(\Omega \setminus B_R(\xi)) \cup (\partial B_R(\xi))$ and, thus, is continuous in $\Omega$.

Recall, that $v \in C^0(\Omega; \mathbb{R})$ is called subharmonic iff

$$v(a) \leq \int_{\partial B_R(x)} u(x) \, dA(x) \quad \forall \ a \in \Omega, \forall \ R > 0, \text{s.t. } B_R(a) \subset \subset \Omega.$$

**Theorem 3.36** (Perron solution). Let $\Omega \subset \mathbb{R}^n$ be a bounded, $\varphi$ be a bounded function on $\partial \Omega$, and denote by

$$S_{\varphi} := \{ v \in C^0(\Omega; \mathbb{R}) : v \text{ is subharmonic and } v \leq \varphi \text{ on } \partial \Omega \}.$$

Then the function

$$u(x) := \sup_{v \in S_{\varphi}} v(x)$$

is harmonic in $\Omega$. This $u$ is called the Perron solution.
Proof. First, we show that this function is well-defined:

- The constant function \( m \equiv \min_{\partial \Omega} \varphi \in S_\varphi \neq \emptyset \).

- We have for all \( v \in S_\varphi \):
  \[
  v(x) \leq \max_{\partial \Omega} v \leq \max_{\partial \Omega} \varphi \leq M < \infty \quad \forall \, x \in \Omega,
  \]
  so that the supremum is well-defined at every point.

Let now \( \xi \in \Omega \) be arbitrary.

- By the definition of the supremum there exists a sequence \( \{v_k\}_{k \in \mathbb{N}} \subset S_\varphi \) such that \( \lim_{k \to \infty} v_k(\xi) = u(\xi) \).
  Without loss of generality we assume \( \{v_k\}_{k \in \mathbb{N}} \) to be monotone increasing and bounded from below, otherwise consider the function
  \[
  w_k(x) := \max\{v_1(x), \ldots, v_k(x), \min_{\partial \Omega} \varphi \}_{v_1 = v_0}
  \]
  we have \( w_k \in C^0, w_l \leq \varphi \) on \( \partial \Omega \) and
  \[
  w_k(a) \leq \int_{\partial B(a)} v_l(x) \, dx \leq \int_{\partial B(a)} w_k(x) \, dx.
  \]

- Let \( R > 0 \) be such that \( B_R(\xi) \subset \subset \Omega \). Then the sequence \( \{P_B v_k\}_{k \in \mathbb{N}} \) of harmonic liftings of \( v_k \) in \( B_R(\xi) \) is also monotone increasing and since \( v_k \leq P_B v_k \) we also have
  \[
  \lim_{k \to \infty} P_B v_k(\xi) = u(\xi).
  \]
  Thus, by Harnack’s convergence theorem (Theorem 3.15) we conclude that \( \{P_B v_k\}_{k \in \mathbb{N}} \) converges uniformly on any \( \Omega' \subset \subset B_R(\xi) \) to a function \( w \) that is harmonic in \( B_R(\xi) \).

- Next we show, that \( w = u \) in \( B_R(\xi) \):
  Clearly we have \( w \leq u \) in \( B_R(\xi) \) and \( w(\xi) = u(\xi) \). If there exists an \( \eta \in B_R(\xi) \) s.t. \( w(\eta) < u(\eta) \), then, since \( u = \sup_{v \in S_\varphi} \), there exists a \( \tilde{v} \in S_\varphi \) s.t. \( w(\eta) < \tilde{v}(\eta) < u(\eta) \). Consider \( W_k := \max\{\tilde{v}, v_k\} \), as before the harmonic liftings \( P_B W_k \) converge uniformly on any \( \Omega' \subset \subset B_R(\xi) \) to a function \( W \) that is harmonic in \( B_R(\xi) \). Moreover, we have
  \[
  w \leq W \leq u \quad \text{and} \quad w(\xi) = W(\xi) = u(\xi).
  \]
  Hence, \( w - W \) is a non-positive harmonic function in \( B_R(\xi) \) with \( (w - W)(\xi) = 0 \), i.e. its maximum is attained in an interior point, and by the maximum principle we conclude \( w - W \equiv 0 \) in contrast to \( w(\eta) < \tilde{v}(\eta) \leq W(\eta) \).

\( \square \)

Remaining question: Is the boundary condition satisfied, i.e. does it hold:

\[
\lim_{a \to b} u(a) = \varphi(b) \quad \forall \, b \in \partial \Omega ?
\]

In general: No!

Counterexample: Take \( \Omega := B_1(0) \setminus \{0\} \). Then \( \partial \Omega = \partial B_1(0) \cup \{0\} \). Consider

\[
\varphi(b) := \begin{cases} 
0, & b \in \partial B_1(0), \\
2, & b = 0,
\end{cases}
\]

we have \( \varphi \in C^0(\partial B_1(0) \cup \{0\}; \mathbb{R}) \). As exercise show that the Perron solution is given by

\[
u(x) = \begin{cases} 
0, & x \in B_1(0) \setminus \{0\}, \\
2, & x = 0,
\end{cases} \]

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and note that \( u \notin C^0(\overline{\Omega}; \mathbb{R}) \). Indeed, there does not exist a harmonic solution to this Dirichlet problem in \( C^2(\Omega; \mathbb{R}) \cap C^0(\overline{\Omega}; \mathbb{R}) \): By the maximum principle for harmonic functions we would have

\[
0 \leq u(x) \leq 2 \quad \forall \ x \in \overline{\Omega} = \overline{B_1(0)},
\]

meaning that \( u \) is bounded \( \Rightarrow \) \( u \) should have a removable isolated singularity in 0, i.e. there exists an extension \( \overline{u} \in C^2(B_1(0); \mathbb{R}) \) such that \( \Delta \overline{u} \equiv 0 \) and \( \overline{u}_{|B_1(0)\setminus\{0\}} \equiv u \). Since in particular \( \overline{u}_{|\partial B_1(0)} \equiv 0 \) we conclude by the maximum principle \( \overline{u} \equiv 0 \in \) the whole \( B_1(0) \) and especially \( \overline{u}(0) = 0 \neq 2 \).

**Definition** (Barrier function). Let \( b \in \partial \Omega \). A function \( w \in C^0(\overline{\Omega}; \mathbb{R}) \) is called a barrier function at \( b \) relative to \( \Omega \) if

(i) \( w \) is subharmonic in \( \Omega \),

(ii) \( w(b) = 0 \),

(iii) \( w(a) < 0 \) for all \( a \in \partial \Omega \setminus \{b\} \).

**Remark 3.37.** 1) From the maximum principle for subharmonic functions it follows that

\[
w(a) < 0 \quad \forall \ a \in \overline{\Omega} \setminus \{b\}.
\]

2) The barrier concept is a local property of the boundary, so that it is also possible just to consider local barriers.

**Lemma 3.38.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, \( \varphi \) be bounded on \( \partial \Omega \) and \( u \) be the Perron solution defined in Theorem 3.36. If there exists a barrier at \( b \in \partial \Omega \) and \( \varphi \) is continuous in \( b \), then

\[
\lim_{a \to b \atop a \in \overline{\Omega}} u(a) = \varphi(b).
\]

**Proof.** Let us denote by \( w \) the barrier at \( b \). Then by the continuity of \( \varphi \) and \( w \) in \( b \) we have

\[
\forall \ \varepsilon > 0 \ \exists \ \delta > 0 : |\varphi(a) - \varphi(b)| < \varepsilon \quad \forall |a - b| < \delta \quad \text{and} \quad |w(a)| < \varepsilon \quad \forall |a - b| < \delta
\]

(3.14a)

since \( w(b) = 0 \). Moreover, \( w(a) < 0 \) for all \( a \neq b \) and again by continuity of \( w \) we have

\[
\forall \ M > 0 \ \exists \ k \geq 0 : -k w(a) \geq 2M \quad \forall |a - b| \geq \delta,
\]

(3.14b)

and in particular for the choice \( M := \sup_{\partial \Omega} |\varphi| < \infty \).

Consider now for \( x \in \overline{\Omega} \) the function

\[
\psi(x) := \varphi(b) - \varepsilon + k w(x)
\]

then for \( x \in \partial \Omega \) if \( |x - b| < \delta \)

\[
\psi(x) = \varphi(b) - \varepsilon + k \sum_{\geq 0} w(x) \leq \varphi(b) - \varepsilon \leq \varphi(x)
\]

(3.14a)

and if \( |x - b| \geq \delta \)

\[
\psi(x)^{(3.14c)} = \varphi(b) - \varepsilon - 2M \leq -M - \varepsilon \leq \varphi(x),
\]

meaning that \( \psi(x) \leq \varphi(x) \) for all \( x \in \partial \Omega \) and since \( \psi \) is subharmonic we obtain \( \psi \in S_{\varphi} \). Thus, for the Perron solution \( u \) we have

\[
\psi(a) \leq u(a) \quad \forall \ a \in \Omega.
\]

Now consider the function

\[
\vartheta(x) := \varphi(b) + \varepsilon - k w(x).
\]

This \( \vartheta \) is superharmonic \((-k \leq 0 \text{ and } \psi \text{ is subharmonic}) \) and as above we deduce

\[
\vartheta(x) \geq \varphi(x) \quad \forall \ x \in \partial \Omega.
\]
For any \( v \in S_\phi \) we have that \( v - \overline{\varphi} \) is subharmonic and moreover: \( \forall x \in \partial \Omega : \ v(x) - \overline{\varphi}(x) \leq \varphi(x) - \varphi(x) = 0, \) so that by the maximum principle for subharmonic functions it follows

\[
v - \overline{\varphi} \leq 0 \quad \text{in } \Omega,
\]
i.e. \( v(a) \leq \overline{\varphi}(a) \) for all \( v \in S_\phi \) and all \( a \in \Omega \). Taking the supremum with respect to \( v \in S_\phi \) we conclude \( u(a) \leq \overline{\varphi}(a) \).

Summarizing, we have

\[
\overline{\varphi}(b) - \epsilon + k w(a) \leq \overline{\varphi}(a) \leq \varphi(b) + \epsilon - k w(a) \quad \forall a \in \Omega.
\]

Hence,

\[
|u(a) - \varphi(b)| \leq \epsilon - k w(a) \leq \epsilon + k \epsilon \quad \forall |a - b| < \delta.
\]

All in all, we have shown:

**Theorem 3.39** (Existence and uniqueness of solutions; Perron’s method). Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain and \( \varphi \in C^0(\partial \Omega; \mathbb{R}) \). If for each \( b \in \partial \Omega \) there exists a barrier function \( w \) at \( b \), then there exists a unique solution \( u \in C^2(\Omega; \mathbb{R}) \cap C^0(\Omega; \mathbb{R}) \) of the boundary value problem

\[
\begin{cases}
\Delta u \equiv 0, & \text{in } \Omega, \\
u = \varphi, & \text{on } \partial \Omega.
\end{cases}
\]

**Question:** When do barrier functions exist?

Sufficient conditions are

(a) If \( \Omega \) is strictly convex, i.e. for all \( b \in \partial \Omega \) there exists a hyperplane, which intersects \( \overline{\Omega} \) only in \( b \). Let \( \eta \in \mathbb{R}^n \setminus \{0\} \) be orthogonal to the hyperplane and pointing away from \( \Omega \). Then, a barrier function at \( b \) is given by

\[
w(x) := \langle \eta, x - b \rangle.
\]

Indeed, \( w \) is linear, \( \Delta w \equiv 0, w(b) = 0 \) and \( w(x) < 0 \) for all \( x \in \partial \Omega \setminus \{0\} \).

(b) If \( \Omega \) satisfies the exterior sphere condition, i.e. for all \( b \in \partial \Omega \) there exists a ball \( \overline{B_R(y)} \) which intersects \( \overline{\Omega} \) only in \( b \). A barrier function at \( b \) is then given by

\[
w(x) := \begin{cases}
- \ln \frac{|x - y|}{R}, & n = 2, \\
- \frac{1}{R^{n-2}} + \frac{1}{|x - y|^{n-2}}, & n \geq 3.
\end{cases}
\]

(c) If \( \Omega \) satisfies the exterior cone condition, i.e. for all \( b \in \partial \Omega \) there exists a cone \( K \) with vertex in \( b \) which intersects \( \overline{\Omega} \) only in \( b \). (See exercise in 2d.)

![Figure 11: Domains that allow for barrier functions.](image)
3.9 Poisson’s equation

Let \( \varphi \in C^0(\partial \Omega; \mathbb{R}) \), \( f \in C^0(\Omega; \mathbb{R}) \). Does there exist a solution \( u \in C^2(\Omega; \mathbb{R}) \cap C^0(\overline{\Omega}; \mathbb{R}) \) of

\[
\begin{align*}
\Delta u &= f, \quad \text{in } \Omega, \\
u &= \varphi, \quad \text{on } \partial \Omega.
\end{align*}
\]

Example 3.1. We start with

\[
\begin{align*}
\Delta u &\equiv -1, \quad \text{in } \Omega, \\
u &= \varphi, \quad \text{on } \partial \Omega.
\end{align*}
\] (3.15)

and consider the function \( v(x) = -\frac{|x|^2}{2n} \). Then \( \nabla v = -\frac{x}{n} \) and \( \Delta v = -1 \). Thus, the function \( w(x) := u(x) - v(x) \) fulfills

\[
\begin{align*}
\Delta w &= 0, \quad \text{in } \Omega, \\
w(x) &= \varphi(x) - \frac{|x|^2}{2n}, \quad \text{on } \partial \Omega.
\end{align*}
\] (3.16)

And, if for all \( b \in \partial \Omega \) there exists a barrier function at \( b \), then by Theorem 3.39 the Dirichlet problem (3.16) is solvable and hence also (3.15).

Question: Given \( f \in C^0(\Omega; \mathbb{R}) \), does a solution of \( \Delta u = f \) exist?

Definition (Newtonian potential). Let \( f \) be integrable on a domain \( \Omega \). Then

\[
N_f(a) := \int_{\Omega} \Psi_a(|x-a|) f(x) \, dx \quad \forall \, a \in \mathbb{R}^n
\]

is called the Newtonian potential of \( f \).

Theorem 3.40. Let \( f \) be bounded and integrable on a bounded domain \( \Omega \). Then
i) \( N_f \in C^\infty(\mathbb{R}^n \setminus \overline{\Omega}; \mathbb{R}) \) and \( \Delta N_f \equiv 0 \) in \( \mathbb{R}^n \setminus \overline{\Omega} \).
ii) \( N_f \in C^1(\mathbb{R}^n; \mathbb{R}) \) and

\[
\nabla_a N_f(a) = \int_{\Omega} \nabla_a [\Psi_a(|x-a|)] f(x) \, dx.
\]

Proof. Ad i): For \( a \in \mathbb{R}^n \setminus \overline{\Omega} \), the fundamental solution \( \Psi_a(\cdot) \) has no singularity in \( \Omega \). Thus, differentiation with respect to the variable \( a \) is allowed due to smoothness and compactness reasons and we obtain

\[
\Delta_a N_f(a) = \int_{\Omega} \Delta_a [\Psi_a(|x-a|)] f(x) \, dx = 0 \quad \forall \, a \in \mathbb{R}^n \setminus \overline{\Omega}.
\]

Ad ii): By assumption we have \( |f| \leq M \), and for all \( a \in \overline{\Omega} \) there exists an \( R > 0 \) such that \( \Omega \subset B_R(a) \). Then,

\[
|N_f(a)| \leq M \int_{B_R(a)} |\Psi_a(|x-a|)| \, dx,
\]

and since the last integral exists by Proposition 3.3 vi) the Newtonian potential is everywhere well-defined. Furthermore, we have

\[
\left| \int_{\Omega} \nabla_a [\Psi_a(|x-a|)] f(x) \, dx \right| \leq M \int_{B_R(a)} \left| \nabla_a [\Psi_a(|x-a|)] \right| \, dx \leq M \int_{\text{prop. 3.3 vi)} \frac{1}{|x-a|^{n-1}}} \, dx = M R,
\]

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so that the function

\[ w(a) := \int_{\Omega} \nabla_a (\Psi_a(|x - a|)) f(x) \, dx \]

is well-defined for all \( a \in \mathbb{R}^n \) and we aim to show that \( w(a) = \nabla_a N_f(a) \) for all \( a \in \mathbb{R}^n \). To this end, consider a function \( \eta \in C^1(\mathbb{R}; \mathbb{R}) \) satisfying

- \( 0 \leq \eta(t) \leq 1 \) for all \( t \in \mathbb{R} \),
- \( \eta(t) = 0 \) for \( t \leq 1 \),
- \( \eta(t) = 1 \) for \( t \geq 2 \),
- \( 0 \leq \eta'(t) \leq 2 \) for all \( t \in \mathbb{R} \)

and define for \( \varepsilon > 0 \)

\[ N_f^\varepsilon(a) := \int_{\Omega} \Psi_a(|x - a|) \eta \left( \frac{|x - a|}{\varepsilon} \right) f(x) \, dx = \int_{\Omega \setminus B_\varepsilon(a)} \Psi_a(|x - a|) \eta \left( \frac{|x - a|}{\varepsilon} \right) f(x) \, dx, \]

so that \( N_f^\varepsilon \in C^1(\mathbb{R}^n; \mathbb{R}^n) \). Furthermore,

\[
|N_f(a) - N_f^\varepsilon(a)| \leq M \int_{B_2(a)} (1 - \eta \left( \frac{|x - a|}{\varepsilon} \right)) |\Psi_a(|x - a|)| \, dx \leq M \int_{B_2(a)} |\Psi_a(|x - a|)| \, dx
\]

like in Prop. 3.3 vi)

\[
\leq \begin{cases}
M \int_0^{2\varepsilon} \ln r \, dr, & n = 2, \\
\frac{M}{n-2} \int_0^{2\varepsilon} r \, dr, & n \geq 3,
\end{cases}
\]

\[
= \begin{cases}
\frac{2M \varepsilon^2 |1 - \ln(4\varepsilon^2)|}{n-2}, & n = 2, \\
\frac{2M \varepsilon^2}{n-2}, & n \geq 3,
\end{cases}
\]

\[ \xrightarrow{\varepsilon \to 0} 0, \]

i.e., \( N_f^\varepsilon \xrightarrow{\varepsilon \to 0} N_f \) (meaning that \( N_f^\varepsilon \) converges uniformly on \( \mathbb{R}^n \) to \( N_f \)). Moreover,

\[
|w(a) - \nabla_a N_f^\varepsilon(a)| \leq M \int_{B_2(a)} \left| \nabla \left[ \Psi_a(|x - a|) \left( 1 - \eta \left( \frac{x - a}{\varepsilon} \right) \right) \right] \right| \, dx
\]

\[
\leq \begin{cases}
2M \varepsilon (1 + \varepsilon |1 - \ln(4\varepsilon^2)|), & n = 2, \\
2M \varepsilon \left( 1 + \frac{2}{n-2} \right), & n \geq 3
\end{cases}
\]

\[ \xrightarrow{\varepsilon \to 0} 0, \]

meaning that \( \nabla N_f^\varepsilon \xrightarrow{\varepsilon \to 0} w \), hence, \( N_f \in C^1(\mathbb{R}^n; \mathbb{R}) \) and \( \nabla N_f = w \).

\[ \square \]

**Problem:**
If \( f \) is merely assumed continuous, the Newtonian potential is not necessarily twice differentiable, cf. \( |D^2 \Psi_a| \lesssim |x - a|^{-n} \).
\( \rightsquigarrow \) need more regularity on \( f \).

**Definition (Hölder continuous).** A function \( f : \Omega \to \mathbb{R} \) is called Hölder-continuous in \( \Omega \) iff there exist constants \( c > 0 \) and \( \alpha > 0 \) such that

\[ |f(x) - f(y)| \leq c |x - y|^\alpha \quad \forall x, y \in \Omega. \]
Remark 3.41. 1) If $\alpha > 1$, then $f$ is constant on any connected component. (Exercise)

2) If $\alpha = 1$, then $f$ is called to be \textit{Lipschitz continuous}.

3) $\alpha \in (0, 1)$ is called the \textit{Hölder exponent}. We denote by $C^{0, \alpha}(\Omega; \mathbb{R})$ the class of all Hölder continuous functions on $\Omega$ with exponent $\alpha$ (the so called Hölder space).

4) We have the following implications:

\[
C^1 \quad \Rightarrow \quad \text{Lipschitz continuous} \quad \Rightarrow \quad \text{Hölder continuous} \quad \Rightarrow \quad \text{uniform continuous} \quad \Rightarrow \quad C^0.
\]

\textbf{Theorem 3.42.} Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $f \in C^{0, \alpha}(\Omega; \mathbb{R})$ (with $\alpha \in (0, 1)$). Then $N_f \in C^2(\Omega; \mathbb{R})$ and for all $a \in \Omega$:

\[
D_a \nabla_a N_f(a) = \int_{\Omega} \left[ D_a \nabla_a (\Psi_a(|x-a|)) \right] (f(x) - f(a)) \, dx - f(a) \int_{\partial\Omega} \nabla_a (\Psi_a(|x-a|)) \nu^\top(x) \, dA(x),
\]

and, in particular, $\Delta N_f = f$ in $\Omega$.

\textit{Proof.} For $a \in \Omega$, let $R > 0$ be such that $\Omega \subset B_R(a)$, then

\[
\int_{\Omega} \left| D_a \nabla_a (\Psi_a(|x-a|)) \right| \frac{|f(x) - f(a)|}{|x-a|^{\alpha-n}} \, dx \leq \frac{c}{nV_n} \int_{B_{R}(a)} |x-a|^{\alpha-n} \, dx = c \int_0^R r^{\alpha-n} \, dr \leq \frac{c}{\alpha} R^\alpha
\]

and since the gradient of $\Psi_a$ is integrable (see previous proof) we obtain that the function

\[
v(a) := \int_{\Omega} \left[ D_a \nabla_a (\Psi_a(|x-a|)) \right] (f(x) - f(a)) \, dx - f(a) \int_{\partial\Omega} \nabla_a (\Psi_a(|x-a|)) \nu^\top(x) \, dA(x)
\]

is well-defined for all $a \in \Omega$. In the previous proof we set $w(a) = \nabla_a N_f(a)$ and consider now

\[
w^{\varepsilon}(a) := \int_{\Omega} \left[ \nabla_a (\Psi_a(|x-a|)) \right] \eta\left(\frac{|x-a|}{\varepsilon}\right) f(x) \, dx
\]

where $\eta(\cdot)$ is the function introduced in the previous proof. Clearly, $w^{\varepsilon} \in C^1(\Omega; \mathbb{R}^n)$, and, differentiating, we obtain

\[
D_a w^{\varepsilon}(a) = \int_{\Omega} D_a \left\{ \left[ \nabla_a (\Psi_a(|x-a|)) \right] \eta\left(\frac{|x-a|}{\varepsilon}\right) \right\} f(x) \, dx
\]

\[
= \int_{\Omega} D_a \left\{ \left[ \nabla_a (\Psi_a(|x-a|)) \right] \eta\left(\frac{|x-a|}{\varepsilon}\right) \right\} (f(x) - f(a)) \, dx
\]

\[
+ f(a) \int_{\Omega} D_a \left\{ \left[ \nabla_a (\Psi_a(|x-a|)) \right] \eta\left(\frac{|x-a|}{\varepsilon}\right) \right\} \, dx
\]

\[
= -D_a \left\{ \left[ \nabla_a (\Psi_a(|x-a|)) \right] \eta\left(\frac{|x-a|}{\varepsilon}\right) \right\} f(a)\int_{\Omega} \nabla_a (\Psi_a(|x-a|)) \eta\left(\frac{|x-a|}{\varepsilon}\right) \nu^\top(x) \, dA(x)
\]

\textit{fund. thm calc.}

\[
\epsilon \geq 0 \quad \int_{\Omega} D_a \left\{ \left[ \nabla_a (\Psi_a(|x-a|)) \right] \eta\left(\frac{|x-a|}{\varepsilon}\right) \right\} (f(x) - f(a)) \, dx - f(a) \int_{\partial\Omega} \nabla_a (\Psi_a(|x-a|)) \nu^\top(x) \, dA(x).
\]
Hence, by subtraction

\[ |v(a) - D_a w^\varepsilon(a)| \leq \left| \int_{B_{2\varepsilon}(a)} D_a \left\{ \left( 1 - \eta \left( \frac{|x - a|}{\varepsilon} \right) \right) \nabla_a (\Psi_a(|x - a|)) \right\} (f(x) - f(a)) \, dx \right| \]

for \( f \in C^{0,\alpha}[0,1] \) with \( |\eta| \leq 2 \)

\[ \leq c \int_{B_{2\varepsilon}(a)} \left( \frac{2}{\varepsilon} \left| \nabla_a \Psi_a \right| |x - a|^\alpha \right) \, dx \]

\[ \leq c \left( \frac{4}{\alpha + 1} + \frac{1}{\alpha} \right) (2\varepsilon)^\alpha \xrightarrow{\varepsilon \to 0} 0, \]

meaning that \( Dw^\varepsilon \xrightarrow{\varepsilon \to 0} v \), since also \( w^\varepsilon \xrightarrow{\varepsilon \to 0} w = \nabla N_f \) we conclude \( N_f \in C^2(\Omega; \mathbb{R}) \) and \( D\nabla N_f = v \).

Extending \( f \) by zero outside \( \Omega \), we obtain for \( B_R(a) \supset \Omega \) in particular for the Laplacian (from above, fundamental thm of calc)

\[ \Delta_a N_f(a) = \int_{B_R(a)} \Delta_a \left( \Psi_a(|x - a|) \right) (f(x) - f(a)) \, dx - f(a) \int_{\partial B_R(a)} \langle \nabla_a \Psi_a(|x - a|), \nu(x) \rangle \, dA(x) \]

Prop 3.3 \( \Rightarrow \)

\[ \int_{\partial B_R(a)} \left( \frac{1}{nV_{\mathbb{R}^n}} \right) \, dA(x) = f(a). \]

\[ \Delta N_f = \begin{cases} f, & \text{in } \Omega, \\ 0, & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \]

i.e., the second derivative jumps along the boundary \( \partial \Omega \) ! (Take as example \( f \equiv -1 \)).

2) Bounded Lipschitz domains satisfy an exterior cone condition, so that a (local) barrier exists at every boundary point.

Summarizing our results from this section we conclude

**Theorem 3.44.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain, \( f \in C^{0,\alpha}[\Omega; \mathbb{R}] \) then we have gained for the Newtonian potential

\[ N_f \in C^1(\mathbb{R}^n; \mathbb{R}) \cap C^\infty(\mathbb{R}^n \setminus \overline{\Omega}; \mathbb{R}) \cap C^2(\Omega; \mathbb{R}) \]

and moreover

\[ \Delta N_f = \begin{cases} f, & \text{in } \Omega, \\ 0, & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \]

**4 Heat equation**

**4.1 Physical interpretation**

Consider the parabolic PDE \( u_t - \Delta_x u = 0 \) for all \( t \in I \subseteq \mathbb{R} \) and all \( x \in \Omega \subseteq \mathbb{R}^n \) with \( u : I \times \Omega \rightarrow \mathbb{R}, (t, x) \mapsto u(t, x) \). Here, \( t \) is called the time variable and \( x \) is called the spatial variable (lat.: spatum \( \equiv \) space).

The heat equation describes the flow/evolution of (e.g.) the heat in a homogeneous and isotropic medium.
Since we have set \( v = \lambda u \), and also changing the length unit, we obtain that the function \( w(t, x) := u(t, \sqrt{\alpha} x) \) satisfies \( w_t = \Delta_x w \). Hence, w.l.o.g. set \( \alpha = 1 \) and consider the heat equation \( u_t = \Delta_x u \).

Recall that \( \Delta_x u \) "describes" the difference between the mean value of \( u \) around a point and the value at this point. By the second law of thermodynamics we know that heat flows from hotter materials to cooler materials. Hence, \( u_t = \Delta_x u \) "says" the rate at which the temperature \( u \) rises at a point is proportional to how much hotter the surrounding material is.

### 4.2 Fundamental solution

Let \( u \) be a solution of

\[
u_t - \Delta_x u \equiv 0 \quad \text{in} \quad (0, \infty) \times \mathbb{R}^n,\]

then, the function \( \bar{u}(t, x) = \mu u(\lambda t, \sqrt{\lambda} x) \) also solves the heat equation for any \( \mu \in \mathbb{R} \) and any \( \lambda > 0 \). Indeed, we have

\[
\bar{u}_t = \lambda \mu \nu_t(\lambda t, \sqrt{\lambda} x)
\]

and \( \Delta_x \bar{u} = \mu \sqrt{\lambda} \Delta_x u(\lambda t, \sqrt{\lambda} x) \)

and the right-hand sides coincide. Thus, we search for (fundamental) solutions that are invariant under a dilation scaling

\[
u(x, t) = \lambda^n u(\lambda t, \sqrt{\lambda} x).
\]

Let \( \lambda = \frac{1}{t} \) (\( t > 0 \)), then

\[
u(x, t) = \frac{1}{t^n} u \left( \frac{1}{\sqrt{t}}, \frac{x}{\sqrt{t}} \right) = \frac{1}{t^n} v \left( \frac{x}{\sqrt{t}} \right)
\]

where we have set \( v(y) := u(1, y) \). For the derivatives we then obtain:

\[
\frac{\partial u}{\partial t} = -\alpha \frac{1}{t^{n+1}} v \left( \frac{x}{\sqrt{t}} \right) + \frac{1}{t^n} (D_y v) \left( \frac{x}{\sqrt{t}} \right) \left( -\frac{1}{2} \right) \frac{x}{t^{n/2}}
\]

and

\[
-\Delta_x u = \frac{1}{t^{n+1}} (\Delta_y v) \left( \frac{x}{\sqrt{t}} \right).
\]

Since \( u \) fulfills \( u_t - \Delta_x u \equiv 0 \) we obtain (for \( y := \frac{x}{\sqrt{t}} \)):

\[
-\frac{1}{t^{n+1}} \left( \alpha v(y) + \frac{1}{2} Dv(y) y + \Delta v(y) \right) \equiv 0.
\]

Assume moreover \( v \) to be radial symmetric, i.e. there exists a function \( w : [0, \infty) \rightarrow \mathbb{R} \) such that \( v(y) = w(|y|) \). Then

\[
\nabla v(y) = w'(|y|) \frac{y}{|y|}
\]

\[
D \nabla v(y) = w''(|y|) \frac{y}{|y|} \otimes \frac{y}{|y|} + w'(|y|) \left\{ \frac{1}{|y|} E_n - y \otimes \frac{y}{|y|^2} \right\}
\]

\[
\Rightarrow \Delta v(y) = w''(|y|) + \frac{n-1}{|y|} w'(|y|)
\]
and \( w \) should satisfy (for \( r := |y| \)) the ODE:

\[
\alpha w(r) + \frac{1}{2} w'(r) r + w''(r) + \frac{n-1}{r} w'(r) \equiv 0,
\]

\( r = \frac{1}{2} r^{1-2\alpha} (r^{2\alpha} w(r))' \),

\( r = r^{1-n} (r^{n-1} w'(r))' \).

Take \( \alpha = \frac{n}{2} \) so that the last equation simplifies to

\[
\frac{d}{dr} \left\{ \frac{1}{2} r^n w(r) + r^{n-1} w'(r) \right\} \equiv 0 \Rightarrow \frac{1}{2} r^n w(r) + r^{n-1} w'(r) \equiv \text{const.}
\]

Take this constant to be zero (possible if we assume \( \lim_{r \to \infty} w(r) = 0 \) and \( \lim_{r \to \infty} w'(r) = 0 \)). Hence, we obtain

\[
w'(r) = -\frac{1}{2} r w(r) \quad \Rightarrow \quad w(r) = c e^{-\frac{|r|^2}{4t}} \text{ for some constant } c \in \mathbb{R}.
\]

Thus, the function

\[
u(t, x) = \frac{c}{t^{n/2}} e^{-\frac{|y|^2}{4t}}
\]
solves the heat equation in \((0, \infty) \times \mathbb{R}^n\).

**Choice of the constant \( c \):** For fixed \( t > 0 \) consider

\[
\frac{c}{t^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4t}} \, dx = \left\{ \begin{array}{l}
y = \frac{x}{2\sqrt{t}} \\
dx = \frac{dx}{(2\sqrt{t})^n}
\end{array} \right\} = c 2^n \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4t}} \, dy \quad \text{Riemann}
\]

\[
c 2^n \left( \int_{-\infty}^\infty e^{-s^2} \, ds \right)^n = c (4\pi)^{n/2} \quad \Rightarrow \quad \text{take } c := \frac{1}{(4\pi)^{n/2}}
\]

**Definition (Fundamental solution).** The function \( \Phi : (0, \infty) \times \mathbb{R}^n \to \mathbb{R} \) with

\[
\Phi(t, x) := \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}
\]
is called fundamental solution of the homogeneous heat equation.

For the integral of the fundamental solution we have shown:

**Lemma 4.1.** At each time \( t > 0 \) it holds:

\[
\int_{\mathbb{R}^n} \Phi(t, x) \, dx = 1.
\]

**4.3 Homogeneous initial value problem**

**Theorem 4.2.** Let \( g \in C^0(\mathbb{R}^n, \mathbb{R}) \) be bounded. Then we have for the function

\[
u(t, x) := \int_{\mathbb{R}^n} \Phi(t, x - y) g(y) \, dy = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y) \, dy
\]

\( i) \ u \in C^\infty((0, \infty) \times \mathbb{R}^n; \mathbb{R}), \)

\( ii) \ \nu_t - \Delta_x u \equiv 0 \text{ in } (0, \infty) \times \mathbb{R}^n, \)

\( iii) \ \lim_{(t,x) \to (0,b)} u(t,x) = g(b) \text{ for all } b \in \mathbb{R}^n. \)
Remark 4.3. ad ii) and iii): This means that the convolution \( u(t, \cdot) = \Phi(t, \cdot) * g \):

\[
\begin{align*}
  u(t, x) &= \int_{\mathbb{R}^n} \Phi(t, x - y)g(y) \, dy = \left\{ \begin{array}{l}
    z = x - y \\
    dz = (-1)^n \, dy
  \end{array} \right\} = (-1)^n \int_{\mathbb{R}^n} \Phi(t, z)g(x - z) \, dz = \int_{\mathbb{R}^n} \Phi(t, z)g(x - z) \, dz \\
  &= (g * \Phi(t, \cdot))(x)
\end{align*}
\]

is, indeed, a solution of the homogeneous heat equation in \( (0, \infty) \times \mathbb{R}^n \) and is continuously extendable on \( [0, \infty) \times \mathbb{R}^n \) with \( u(0, x) = g(x) \) for all \( x \in \mathbb{R}^n \).

**Proof of Theorem 4.2.** We have \( \Phi \in C^\infty((0, \infty) \times \mathbb{R}^n; \mathbb{R}) \) and for any \( \delta > 0 \) we obtain for all \( t \geq \delta \):

\[
|D^n_{(t,x)} \Phi(t, x)| \leq c(\alpha, \delta) \, |p_2(\alpha)| \leq \tilde{c}(\alpha, \delta) \quad \forall \alpha \in \mathbb{N}_0^n,
\]

where \( p_2(\alpha) \) is a polynomial of degree \( \leq 2 |\alpha| \). Meaning that for all \( \alpha \in \mathbb{N}_0^n \) the \( |\alpha| \)-th derivatives are uniformly bounded.

Hence, \( u \in C^\infty((0, \infty) \times \mathbb{R}^n; \mathbb{R}) \) and

\[
u \_t(t, x) - \Delta_x u(t, x) = \int_{\mathbb{R}^n} \Phi(t, x - y)g(y) \, dy = 0 \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}^n \Rightarrow i) \& ii).
\]

ad iii): For fixed \( b \in \mathbb{R}^n, \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
|g(y) - g(b)| < \varepsilon \quad \forall y \in \mathbb{R}^n : |y - b| < \delta.
\]

Then for all \( x \in \mathbb{R}^n \) such that \( |x - b| < \frac{\delta}{2} \) we obtain

\[
|u(t, x) - g(b)| = \left| \int_{\mathbb{R}^n} \Phi(t, x - y)g(y) \, dy - g(b) \int_{\mathbb{R}^n} \Phi(t, x - y) \, dy \right| \\
\leq \int_{B_\delta(b)} \Phi(t, x - y) |g(y) - g(b)| \, dy + \int_{\mathbb{R}^n \setminus B_\delta(b)} \Phi(t, x - y) |g(y) - g(b)| \, dy \\
\leq \varepsilon \int_{\mathbb{R}^n \setminus B_\delta(b)} \Phi(t, x - y) \, dy + \int_{\mathbb{R}^n \setminus B_\delta(b)} \Phi(t, x - y) |g(y) - g(b)| \, dy \\
= \varepsilon + \int_{\mathbb{R}^n \setminus B_\delta(b)} \Phi(t, x - y) |g(y) - g(b)| \, dy.
\]

In order to estimate the remaining integral, note that for \( |x - b| < \frac{\delta}{2} \) and \( |y - b| \geq \delta \) we obtain:

\[
|y - b| \leq |y - x| + |x - b| < |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2} |y - b| \quad \Rightarrow \quad \frac{1}{2} |y - b| \leq |y - x|,
\]

so that

\[
\int_{\mathbb{R}^n \setminus B_\delta(b)} \Phi(t, x - y) |g(y) - g(b)| \, dy \leq \frac{M}{4\pi t} \int_{\mathbb{R}^n \setminus B_\delta(b)} \Phi(t, x - y) \, dy = \frac{2M}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B_\delta(b)} e^{-\frac{|x-y|^2}{4t}} \, dy \\
\leq \frac{2M}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B_\delta(b)} e^{-\frac{|x-b|^2}{4t}} \, dy = \frac{2M}{(4\pi t)^{n/2}} nV_n \int_{\delta} e^{-\frac{r^2}{4\pi t}} r^{n-1} \, dr.
\]

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Recall the variation of parameters / of constants method to find a particular solution of an inhomogeneous ode.

\[ u(t, x) - \Delta_x u(t, x) = f(t) \quad \forall \ (t, x) \in (0, \infty) \times \mathbb{R}^n, \]

\[ u(0, x) = 0 \quad \forall x \in \mathbb{R}^n. \]  

Summarizing, we have shown, that if \( |x - b| < \frac{\varepsilon}{2} \) and \( t > 0 \) is small enough, then \( |u(t, x) - g(b)| < 2\varepsilon \).

\[ 4.4 \quad \text{Inhomogeneous initial value problem} \]

Recall the variation of parameters / of constants method to find a particular solution of an inhomogeneous ode:

\[ u'(t) + p(t) u(t) = f(t) \quad \text{with} \quad p, f \in \mathcal{C}^0(I; \mathbb{R}). \]

Then

\[ \bar{u}(t) := e^{-\int p(t) \, dt} \]

solves the homogeneous ode \( u'(t) + p(t) u(t) = 0 \) for all \( c \in \mathbb{R} \). Now, search for a particular solution of the inhomogeneous ode using the method variation of parameters in the form

\[ u(t) = c(t) e^{-\int p(t) \, dt} \]

where \( c(\cdot) \) is an unknown function.

\[ \Rightarrow \quad u'(t) = c'(t) e^{-\int p(t) \, dt} - c(t) p(t) e^{-\int p(t) \, dt} \]

substituting into the inhomogeneous equation we have

\[ c'(t) e^{-\int p(t) \, dt} - c(t) p(t) e^{-\int p(t) \, dt} + c(t) p(t) e^{-\int p(t) \, dt} = f(t) \]

\[ \Rightarrow \quad c'(t) = f(t) e^{\int p(t) \, dt} \quad \text{integrate} \quad c(t) = \ldots \quad \leadsto \quad u(t) = \ldots \]

Similar for inhomogeneous pde (here: this method is called Duhamel’s principle): Let’s consider the inhomogeneous heat equation with vanishing initial condition:

\[ \begin{cases} 
  u_t(t, x) - \Delta_x u(t, x) = f(t, x) & \forall (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
  u(0, x) = 0 & \forall x \in \mathbb{R}^n. 
\end{cases} \quad (4.2) \]

For fixed \( s \in (0, t) \) the function

\[ u^s(t, x) := \int_{\mathbb{R}^n} \Phi(t - s, x - y) f(s, y) \, dy \]

solves the shifted homogeneous initial value problem (see Theorem 4.2):

\[ \begin{cases} 
  \frac{\partial}{\partial t} u^s(t, x) - \Delta_x u^s(t, x) = 0 & \text{in} \ (s, \infty) \times \mathbb{R}^n, \\
  u^s(s, x) = f(s, x) & \forall x \in \mathbb{R}^n. 
\end{cases} \]

Duhamel’s idea: Integrate with respect to \( s \) to obtain a solution of (4.2):

\[ u(t, x) = \int_{0}^{t} u^s(t, x) \, ds = \int_{0}^{t} \int_{\mathbb{R}^n} \Phi(t - s, x - y) f(s, y) \, dy \, ds \]

\[ = \int_{0}^{t} \frac{1}{(4\pi(t - s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x - y|^2}{4(t - s)}} f(s, y) \, dy \, ds \quad (4.3) \]
Theorem 4.4. Let $f \in C^2_t((0, \infty) \times \mathbb{R}^n; \mathbb{R}) \cap C^0_t((0, \infty) \times \mathbb{R}^n; \mathbb{R})$ be bounded with bounded derivatives of first and second order. Then, $u$ defined by (4.3) is a solution of (4.2). More precisely,

i) $u \in C^2_t((0, \infty) \times \mathbb{R}^n; \mathbb{R})$,

ii) $u(t, x) - \Delta_x u(t, x) = f(t, x)$ for all $t > 0$ and all $x \in \mathbb{R}^n$,

iii) \[ \lim_{(t,x) \to (0,b)} u(t,x) = 0 \text{ for all } b \in \mathbb{R}^n. \]

Proof. Choose $M > 0$ such that

\[ |f(t,x)| \leq M, \quad |D(t,x)f(t,x)| \leq M, \quad |D_x^2 f(t,x)| \leq M, \]

then:

\[ |u(t,x)| \leq M \int_0^t \int_{\mathbb{R}^n} \Phi(t-s,x-y) \, dy \, ds = t \int_0^t \int_{\mathbb{R}^n} \Phi(t-s,x-y) \, dy \, ds \xrightarrow{t \to 0} 0 \quad \implies \quad iii). \]

ad i): Changing variables we have

\[ u(t, x) = \left\{ \begin{array}{ll} 0 \quad & t = \tau \leq 0 \\ -\int_0^{\tau} \int_{\mathbb{R}^n} \Phi(\tau, z) f(t - \tau, x - z) \, dz \, d\tau = \int_0^{\tau} \int_{\mathbb{R}^n} \Phi(\tau, z) f(t - \tau, x - z) \, dz \, d\tau. \end{array} \right. \]

Since for the integrand we have

\[ |D(t,x) [\Phi(\tau, z) f(t - \tau, x - z)]| \leq M \Phi(\tau, z) \quad \text{and} \quad |D_x^2 [\Phi(\tau, z) f(t - \tau, x - z)]| \leq M \Phi(\tau, z) \quad (4.4) \]

we argue again by Lebesgue’s theorem on bounded convergence to conclude that $u \in C^2_t((0, \infty) \times \mathbb{R}^n; \mathbb{R})$ and to calculate

\[ u_t(t, x) = \int_0^{\tau} \int_{\mathbb{R}^n} \Phi(\tau, z) f_t(t - \tau, x - z) \, dz \, d\tau + \int_{\mathbb{R}^n} \Phi(\tau, z) f(0,x - z) \, dz. \]

Hence:

\[ u_t(t, x) - \Delta_x u(t, x) = \int_0^{\tau} \int_{\mathbb{R}^n} \Phi(\tau, z) \left( \frac{\partial}{\partial \tau} - \Delta_z \right) f(t - \tau, x - z) \, dz \, d\tau + \int_{\mathbb{R}^n} \Phi(\tau, z) f(0,x - z) \, dz \]

\[ = \int_0^{\tau} \int_{\mathbb{R}^n} \Phi(\tau, z) \left( - \frac{\partial}{\partial \tau} - \Delta_z \right) f(t - \tau, x - z) \, dz \, d\tau \]

\[ + \int_{\mathbb{R}^n} \Phi(\tau, z) f(0,x - z) \, dz \]

\[ = I_1 + I_2 + I_3 \quad (4.5) \]

We have:

\[ |I_1| \leq 2M \int_0^{\tau} \int_{\mathbb{R}^n} \Phi(\tau, z) \, dz \, d\tau = 2M \varepsilon \xrightarrow{\varepsilon \to 0} 0 \]

and

\[ I_2 = \int_{\varepsilon}^{\tau} \int_{\mathbb{R}^n} \Phi(\tau, z) \left( - \frac{\partial}{\partial \tau} \right) f(t - \tau, x - z) \, dz \, d\tau + \int_{\varepsilon}^{\tau} \lim_{B \to 0} \int_{B_n(0)} \Phi(\tau, z) \left( - \text{div}_z \nabla_z \right) f(t - \tau, x - z) \, dz \, d\tau \]
Remark

Thus, reinserting in (4.5) we obtain

\[ u(t, x) - \Delta_x u(t, x) = f(t, x) \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}^n, \]

\[ u(0, b) = g(b) \quad \forall b \in \mathbb{R}^n, \quad \text{initial values at time } t = 0. \]

\[ \square \]

**Corollary 4.5.** Let \( g \in C^0(\mathbb{R}^n; \mathbb{R}) \) be bounded and \( f \in C^2((0, \infty) \times \mathbb{R}^n; \mathbb{R}) \cap C^0((0, \infty) \times \mathbb{R}^n; \mathbb{R}) \) be bounded with bounded derivatives of first and second order. Then,

\[ u(t, x) = \int_{\mathbb{R}^n} \Phi(t, x - y) g(y) \, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(t - s, x - y) f(s, y) \, dy \, ds \]

is a solution of

\[
\begin{align*}
\{ u_t(t, x) - \Delta_x u(t, x) &= f(t, x) \quad \forall (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
u(0, b) &= g(b) \quad \forall b \in \mathbb{R}^n, \quad \text{initial values at time } t = 0.
\end{align*}
\]

**4.5 Mean value property**

**Definition** (Parabolic cylinder). Let \( \Omega \subseteq \mathbb{R}^n \) be non-empty and open. For fixed time \( T > 0 \) we define

- the **parabolic cylinder** by
  \[ \Omega_T := (0, T] \times \Omega, \]

- the **parabolic boundary** of \( \Omega_T \) by
  \[ \Gamma_T := \overline{\Omega_T} \setminus \Omega_T = (\{0\} \times \Omega) \cup ([0, T] \times \partial \Omega). \]

**Remark 4.6.** Note that the “final slice” \( \{T\} \times \Omega \) does not belong to the parabolic boundary but to the parabolic interior \( \Omega_T \).
Recall, that closed balls in $\mathbb{R}^n$ can be expressed by the fundamental solution of the Laplace equation:

$$B_r(a) = \{x \in \mathbb{R}^n : |x - a| \leq r\} = \{x \in \mathbb{R}^n : \Psi_a(|x - a|) \leq \Psi_a(r)\}.$$

Note that $\Psi_a(\cdot)$ was monotone increasing and the spheres $\partial B_r(a)$ can be seen as its level sets. Now consider the region in space-time, the boundary of which are level sets of the fundamental solution of the heat equation $\Phi(\vartheta - t, a - x)$:

**Definition (Heat ball).** For fixed $(\vartheta, a) \in (0, \infty) \times \mathbb{R}^n$ and $r > 0$ define the (closed) heat ball with center $(\vartheta, a)$ and radius $r > 0$ by

$$E_r(\vartheta, a) := \{(t, x) \in (0, \infty) \times \mathbb{R}^n : t \leq \vartheta, \Phi(\vartheta - t, a - x) \geq \frac{1}{r^n}\} =\{(t, x) \in (0, \infty) \times \mathbb{R}^n : t \leq \vartheta, \frac{1}{(4\pi(\vartheta - t))^{n/2}} e^{-|a-x|^2/4(\vartheta-t)} \geq \frac{1}{r^n}\} =\{(t, x) \in (0, \infty) \times \mathbb{R}^n : t \leq \vartheta, |a-x|^2 \leq 2n(\vartheta - t) \ln \left(\frac{r^2}{4\pi(\vartheta - t)}\right)\}.$$

How does the heat ball look like?

- Only times before $\vartheta$ matter, no future times.
- The heat ball is invariant under spatial rotations around $a$.
- The “slices” of the heat ball (i.e. at a fixed time) are balls in $\mathbb{R}^n$ with variable radius.
- At time $t = \vartheta - \frac{r^2}{4\pi}$ and $t = \vartheta$ this radius is zero.
- We should have $\vartheta > \frac{r^2}{4\pi}$, then $E_r(\vartheta, a) \subset \subset (0, \infty) \times \mathbb{R}^n$.

Figure 14: Indicated is a heat ball $E_r(\vartheta, a)$ with center $(\vartheta, a)$. 
**Lemma 4.7.** Let \( \theta > \frac{1}{4a} \). Then: \[
\int_{E_1(\theta,0)} \frac{|y|^2}{(\theta - s)^2} \, d(s, y) = 4.
\]

**Proof.** We have
\[
\int_{E_1(\theta,0)} \frac{|y|^2}{(\theta - s)^2} \, d(s, y) = \int_{\theta - \frac{1}{4a} B_\rho(0)} \int_{\theta - \frac{1}{4a} B_\rho(0)} \frac{|y|^2}{(\theta - s)^2} \, dy \, ds,
\]
where
\[
\rho(s) = \sqrt{2n(\theta - s) \ln \left( \frac{1}{4\pi(\theta - s)} \right)}
\]
by the definition of the heat ball. Since
\[
\int_{B_\rho(0)} |y|^2 \, dy = nV_n \int_0^\rho r^{2n-1} \, dr = \frac{nV_n \rho^{n+2}}{n+2}
\]
we conclude
\[
\int_{E_1(\theta,0)} \frac{|y|^2}{(\theta - s)^2} \, d(s, y) = \frac{nV_n}{n+2} \int_{\theta - \frac{1}{4a}}^\theta \left[ 2n(\theta - s) \ln \left( \frac{1}{4\pi(\theta - s)} \right) \right]^{\frac{n+2}{2}} \left( \frac{1}{\theta - s} \right)^{\frac{n+2}{2}} \, ds
\]
\[
= \frac{4\pi(\theta - s)}{e^{-\frac{2\pi^2}{s}}} \left[ \frac{2}{\pi} \right]^{\frac{n+2}{2}} \left[ \frac{2}{\pi} \right]^{\frac{n+2}{2}} \frac{2}{\pi} = \frac{8V_n}{(n+2)\pi^{n/2}} \int_0^\infty e^{-\tau} \frac{n+2}{\pi} \, d\tau
\]
\[
= \frac{8V_n}{(n+2)\pi^{n/2}} \int_0^\infty e^{-\tau} \frac{n+2}{\pi} \, d\tau = \frac{8V_n}{(n+2)\pi^{n/2}} \frac{\Gamma \left( \frac{n+2}{2} \right)}{\Gamma \left( \frac{n+2}{2} + 1 \right)}
\]
\[
= \frac{8V_n}{(n+2)\pi^{n/2}} \frac{\pi^{n/2}}{\Gamma \left( \frac{n+2}{2} + 1 \right)} = 4,
\]
whereby we used that the \( \Gamma \)-function \( \Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t} \, dt \) fulfills \( \Gamma(z+1) = z \Gamma(z) \) and \( V_n = \frac{\pi^{n/2}}{\Gamma \left( \frac{n}{2} + 1 \right)} \). \( \square \)

**Theorem 4.8** (Mean value property for solutions of the heat equation). Let \( u \in C^2_\Gamma(\Omega_T; \mathbb{R}) \) be a solution to the homogeneous heat equation. For \((\theta, a) \in \Omega_T, r < \sqrt{4\pi \theta}\) such that \( E_r(\theta, a) \subset \Omega_T \) we have
\[
u(\theta, a) = \frac{1}{4\pi^2} \int_{E_r(\theta, a)} u(t, x) \frac{|a - x|^2}{(\theta - t)^2} \, dt \, dx.
\]

**Proof.** Idea: Set \( \phi(r) := \frac{1}{4\pi^2} \int_{E_r(\theta, a)} u(t, x) \frac{|a - x|^2}{(\theta - t)^2} \, dt \, dx \) and show \( \phi'(r) \equiv 0 \).

In order to differentiate with respect to \( r \): rescale:
\[
\frac{\theta - t}{r^2} = \theta - s, \quad \frac{a - x}{r} = 0 - y \quad \Rightarrow \quad d(s, y) = \frac{1}{r^{n+2}} \, dt \, dx,
\]
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so that the center of the new heat ball is in \((\theta, 0)\) with radius 1 and

\[
\phi(r) = \frac{1}{4} \int_{E_{r}(\theta, 0)} u(\vartheta - r^{2}(\theta - s), a + ry) \frac{|y|^{2}}{(\theta - s)^{2}} \, d(s, y)
\]

so that by chain rule we obtain

\[
\phi'(r) = \frac{1}{4} \int_{E_{r}(\theta, 0)} \left[ u_{t}(\vartheta - r^{2}(\theta - s), a + ry)(-2r(\theta - s)) + \left( \nabla_{x} u(\vartheta - r^{2}(\theta - s), a + ry), y \right) \right] \frac{|y|^{2}}{(\theta - s)^{2}} \, d(s, y)
\]

\[
= \frac{1}{4r^{n+1}} \int_{E_{r}(\theta, a)} \left[ u_{t}(t, x)(-2(\vartheta - t)) - \left( \nabla_{x} u(t, x), a - x \right) \right] \frac{|a - x|^{2}}{(\vartheta - t)^{2}} \, d(t, x) =: I_{1} + I_{2}.
\]

Introducing the auxiliary function

\[
h(t, x) := -\frac{n}{2} \ln(4\pi(\vartheta - t)) - \frac{|a - x|^{2}}{4(\vartheta - t)} + n \ln r
\]

we have

\[
e^{h(t, x)} = \Phi(\vartheta - t, a - x)r^{n}
\]

and, thus,

\[
h(t, x) \equiv 0 \quad \text{on} \partial E_{r}(\theta, a),
\]

moreover

\[
\nabla_{x} h(t, x) = \frac{a - x}{2(\vartheta - t)}.
\]

so that we can recompose the first integral

\[
I_{1} = \frac{1}{4r^{n+1}} \int_{E_{r}(\theta, a)} -4 u_{t}(t, x) \left( \nabla_{x} h(t, x), a - x \right) \, d(t, x)
\]

\[
= \frac{1}{4r^{n+1}} \int_{\partial - \frac{r^{2}}{16} B_{r}(\theta)(a)} \int_{0}^{\vartheta} -4 u_{t}(t, x) \left( \nabla_{x} h(t, x), a - x \right) \, dx \, dt
\]

int. by parts \( w/ \text{resp. } x \)

\[
h_{|_{\partial E_{r}(\theta, a) \equiv 0}} = \frac{1}{4r^{n+1}} \int_{\partial - \frac{r^{2}}{16} B_{r}(\theta)(a)} \int_{0}^{\vartheta} 4h(t, x) \left( \nabla_{x} u_{t}(t, x), a - x \right) - 4h(t, x) n u_{t}(t, x) \, dx \, dt
\]

\[
= \frac{1}{4r^{n+1}} \int_{E_{r}(\theta, a)} h(t, x) \left( \nabla_{x} u_{t}(t, x), a - x \right) - n h(t, x) u_{t}(t, x) \, d(t, x)
\]

int. by parts \( w/ \text{resp. } t \)

\[
h_{|_{\partial E_{r}(\theta, a) \equiv 0}} = \frac{1}{4r^{n+1}} \int_{E_{r}(\theta, a)} -n \left( \nabla_{x} u(t, x), \nabla_{x} h(t, x) \right) - n h(t, x) u_{t}(t, x) \, d(t, x) + \frac{1}{4r^{n+1}} \int_{E_{r}(\theta, a)} \left( \nabla_{x} u(t, x), a - x \right) \frac{|a - x|^{2}}{(\vartheta - t)^{2}} \, d(t, x)
\]

\[= -I_{2}
\]

int. by parts \( w/ \text{resp. } x \)

\[
h_{|_{\partial E_{r}(\theta, a) \equiv 0}} = \frac{1}{4r^{n+1}} \int_{E_{r}(\theta, a)} n h(t, x) \Delta_{x} u(t, x) - n h(t, x) u_{t}(t, x) \, d(t, x) - I_{2} u_{t} - \frac{\Delta_{x} u \equiv 0}{\equiv 0} - I_{2}.
\]
Thus, \( \phi'(r) = I_1 + I_2 \equiv 0 \) so that \( \phi(r) \equiv \text{const} \) is independent of \( r \) and we conclude

\[
\phi(r) = \lim_{r \to 0} \phi(r) = \lim_{r \to 0} \frac{1}{4\pi r^n} \int_{E_r(\vartheta, a)} u(t, x) \frac{|a - x|^2}{(\vartheta - t)^2} \, d(t, x) = \frac{1}{4\pi} \int_{E_{\vartheta, a}} \frac{|a - x|^2}{(\vartheta - t)^2} \, d(t, x)
\]

\[
\begin{aligned}
\langle \vartheta - t \rangle &= \langle \theta - s \rangle \\
\langle a - x \rangle &= 0 - y
\end{aligned}
\]

\[
= u(\vartheta, a) \lim_{r \to 0} \frac{1}{4} \int_{E_{\vartheta, a}} \frac{|y|^2}{(\theta - s)^2} \, d(s, y) = u(\vartheta, a).
\]

\( \square \)

### 4.6 Maximum principles

**Theorem 4.9** (Strong maximum principle for solutions of the heat equation). Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, \( T > 0 \) and \( u \in C^2(\Omega_T; \mathbb{R}) \cap C^0(\overline{\Omega}; \mathbb{R}) \) be a solution of the homogeneous heat equation. If there exists \((\vartheta, a) \in \Omega_T\) such that

\[
u(a) = \max_{\Omega_T} u
\]

then \( u \equiv \text{const} \) in \( \overline{\Omega_T} \).

**Remark 4.10.**

i) This means: If \( u \) attains its maximum at an interior point of the parabolic cylinder \( \Omega_T \), then \( u \) is constant at all earlier times.

ii) Conclusion is wrong for future times:

**Example:**

\[
u(t, x) := \begin{cases}
0 & \text{in } [0, \vartheta] \times \Omega, \\
-\frac{1}{(4\pi(t-\vartheta))^n} e^{-\frac{|x|^2}{4(t-\vartheta)}} & \text{for } t > \vartheta, x \in \overline{\Omega},
\end{cases}
\]

then \( u \in C^\infty([0, \infty) \times \overline{\Omega}; \mathbb{R}) \) solves the heat equation in \((0, \infty) \times \Omega \). However for all \( T > \vartheta \):

\[
\max_{\Omega_T} u = 0 \quad \text{and} \quad u \equiv 0 \text{ in } [0, \vartheta] \times \overline{\Omega},
\]

but \( u < 0 \) for all \( t > \vartheta \) (i.e. for all future times).

**Proof of Theorem 4.9.** We set \( M := \max_{\Omega_T} u \) and find an \( r > 0 \) such that \( E_r(\vartheta, a) \subset \Omega_T \). Then by the mean value property

\[
M = u(\vartheta, a) \equiv \frac{1}{4\pi r^n} \int_{E_r(\vartheta, a)} u(t, x) \frac{|a - x|^2}{(\vartheta - t)^2} \, d(t, x) \leq \frac{M}{4\pi r^n} \int_{E_r(\vartheta, a)} \frac{|a - x|^2}{(\vartheta - t)^2} \, d(t, x)
\]

\[
\begin{aligned}
\langle \vartheta - t \rangle &= \langle \theta - s \rangle \\
\langle a - x \rangle &= 0 - y
\end{aligned}
\]

\[
= \frac{M}{4} \int_{E_{\vartheta, a}} \frac{|y|^2}{(\theta - s)^2} \, d(s, y) = M,
\]

meaning that \( u \equiv M \) in the whole closed heat ball \( E_{\vartheta, a} \).

Let now \( s \in (0, \vartheta) \) and \( y \in \Omega \). Since \( \Omega \) is connected there exists a path \( \gamma : [0, 1] \to \Omega \) connecting \( \gamma(0) = y \) with \( \gamma(1) = a \). Consider

\[
\xi := \min\{\lambda \in [0, 1] : u((1 - \lambda)s + \lambda \vartheta, \gamma(\lambda)) = M\},
\]

connects \((s, y)\) with \((\vartheta, a)\).

Since \( u \in C^0 \) this minimum is attained. Assume \( \xi > 0 \). Then \( u(t_\xi, x_\xi) = M \) for \( t_\xi := (1 - \xi)s + \xi \vartheta, x_\xi := \gamma(\xi) \) and there exists a radius \( \rho > 0 \) such that

\[
u(t_\xi, x_\xi) = M \text{ in the whole } E_{r}(t_\xi, x_\xi),
\]

meaning that also \( u(t_\xi - \sigma, x_\xi - \sigma) = M \) for some small \( \sigma > 0 \), in contrary to the minimality of \( \xi \). Hence, \( \xi = 0 \), i.e. \( u(s, y) = M \). Since \( s \in (0, \vartheta) \) and \( y \in \Omega \) were arbitrary, we conclude by the continuity of \( u \) that \( u \equiv M \) in \( \overline{\Omega_\vartheta} \). \( \square \)
Remark 4.11. Also the strong minimum principle is valid for solutions of the heat equations, just replace max by min in Theorem 4.9.

**Corollary 4.12 (Weak maximum principle for solutions of the heat equation).** Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $T > 0$ and $u \in C^2_1(\Omega_T; \mathbb{R}) \cap C^0(\overline{\Omega_T}; \mathbb{R})$ be a solution of the homogeneous heat equation. Then

$$\max_{\Omega_T} u = \max_{\Gamma_T} u,$$

meaning that the maximum is attained on the parabolic boundary.

**Proof.** W.l.o.g let $\Omega$ be connected, otherwise argue on a connected component:

- “$\geq$”; $\checkmark$ since $\Gamma_T \subset \Omega_T$,
- “$\leq$”; let $(\vartheta, a) \in \Omega_T$ such that $\max_{\Omega_T} u = u(\vartheta, a)$
  - if $(\vartheta, a) \in \Gamma_T$ $\checkmark$
  - if $(\vartheta, a) \in \Omega_T$ we apply Theorem 4.9 to conclude that $u \equiv \text{const in } \overline{\Omega_\vartheta}$ and especially

$$\max_{\Omega_T} u = u(\vartheta, a) = \max_{\Gamma_T} u \leq \max_{\Gamma_T} u.$$

\[\square\]

**Remark 4.13.** For the weak minimum principle for solutions of the heat equation replace max by min in Corollary 4.12.

**Remark 4.14 (Infinite propagation speed).**

- Recall that for a bounded and continuous function $g$ a solution of the homogeneous initial value problem

$$\begin{cases}
  u_t - \Delta_x u \equiv 0 & \text{in } (0, \infty) \times \mathbb{R}^n, \\
  u(0, b) = g(b) & \forall b \in \mathbb{R}^n,
\end{cases}$$

is given by (see Theorem 4.2):

$$u(t, x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y) \, dy.$$

Thus, if $g \geq 0$ and $g(\hat{b}) > 0$ for some $\hat{b} \in \mathbb{R}^n$, then $u(t, x) > 0$ for all $t > 0$ and all $x \in \mathbb{R}^n$. $\Rightarrow$ heat is transported directly to any point if the initial temperature is non-negative and positive somewhere $\Rightarrow$ infinite propagation speed

- Can also be seen by the strong minimum principle: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $T > 0$ and $u \in C^2_1(\Omega_T; \mathbb{R}) \cap C^0(\overline{\Omega_T}; \mathbb{R})$ be a solution of

$$\begin{cases}
  u_t - \Delta_x u \equiv 0 & \text{in } \Omega_T, \\
  u \equiv 0 & \text{on } [0, T] \times \partial \Omega \quad \text{boundary values,} \\
  u(0, b) = g(b) & \forall b \in \Omega \quad \text{initial values.}
\end{cases}$$

If $g \geq 0$ and $g(\hat{b}) > 0$ for some $\hat{b} \in \Omega$, then $u > 0$ in $\Omega_T$, since otherwise, by the strong minimum principle, we would conclude $u \equiv 0$ in $\Omega_\vartheta$ (for some $\vartheta \leq T$) and especially $0 = u(0, \hat{b}) = g(\hat{b}) > 0$.

**Theorem 4.15 (Uniqueness on bounded domains).** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $T > 0$, $f \in C^0(\Omega_T; \mathbb{R})$, $g \in C^0(\Gamma_T; \mathbb{R})$. Then there exists at most one solution $u \in C^2_1(\Omega_T; \mathbb{R}) \cap C^0(\overline{\Omega_T}; \mathbb{R})$ of the problem

$$\begin{cases}
  u_t - \Delta_x u = f & \text{in } \Omega_T, \\
  u = g & \text{on } \Gamma_T.
\end{cases}$$

**Proof.** Let $u, v$ be two solutions. Consider $w := u - v$. Then $w \in C^2_1(\Omega_T; \mathbb{R}) \cap C^0(\overline{\Omega_T}; \mathbb{R})$ solves

$$\begin{cases}
  w_t - \Delta_x w \equiv 0 & \text{in } \Omega_T, \\
  w \equiv 0 & \text{on } \Gamma_T.
\end{cases}$$

By the weak maximum and minimum principle for solutions of the heat equation we conclude

$$\max_{\Omega_T} w = \max_{\Gamma_T} w = 0 \quad \text{as well as} \quad \min_{\Omega_T} w = \min_{\Gamma_T} w = 0 \quad \Rightarrow \quad w \equiv 0 \text{ in } \overline{\Omega_T}. \quad \square$$
Proof. If so that for some positive constants $v$ be a solution of
\[
(T > 0, g \in C^0(\mathbb{R}^n; \mathbb{R}) \text{ and } u \in C^2_t(\mathbb{R}_T^2; \mathbb{R}) \cap C^0(\mathbb{R}_T^2; \mathbb{R}))
\]
be a solution of
\[
\begin{cases}
  u_t - \Delta_x u = 0 & \text{in } \mathbb{R}_T^2, \\
  u(0, b) = g(b) & \forall b \in \mathbb{R}^n.
\end{cases}
\]
If $u$ satisfies the growth condition
\[
u(t, x) \leq A e^{\alpha|t|^2} \forall t \in [0, T] \forall x \in \mathbb{R}^n,
\]
for some positive constants $A, a > 0$, then
\[
\sup_{t \in \mathbb{R}_T^2} u = \sup_{\mathbb{R}^n} g.
\]

Proof. If $T < \frac{1}{4a}$ there exist $\varepsilon, \gamma > 0$ such that
\[
\frac{1}{4(T + \varepsilon)} = a + \gamma.
\]

For fixed $y \in \mathbb{R}^n, \mu > 0$ consider the function
\[
v(t, x) := u(t, x) - \frac{\mu}{(T + \varepsilon - t)^{\gamma/2}} e^{\frac{|x-y|^2}{4(T + \varepsilon - t)}}.
\]
It follows like in the derivation of the fundamental solution that this $v$ solves the homogeneous heat equation:
\[
v_t - \Delta_x v \equiv 0 \quad \text{in } \mathbb{R}_T^2.
\]
Furthermore, for $b \in \mathbb{R}^n$
\[
v(0, b) = u(0, b) - \frac{\mu}{(T + \varepsilon - t)^{\gamma/2}} e^{\frac{|x-y|^2}{4(T + \varepsilon - t)}} \leq u(0, b) = g(b)
\]
and for $x \in \partial B_r(y)$ and $t \in [0, T]$
\[
v(t, x) \leq u(t, x) - \frac{\mu}{(T + \varepsilon - t)^{\gamma/2}} e^{\frac{|x-y|^2}{4(T + \varepsilon - t)}} \leq A e^{\alpha|t|^2} - \frac{\mu}{(T + \varepsilon - t)^{\gamma/2}} e^{\frac{|x-y|^2}{4(T + \varepsilon - t)}} \leq A e^{\alpha|t|^2} - \mu[4(a + \gamma)]^{\gamma/2} e^{r^2(a + \gamma)} \xrightarrow{\gamma \to \infty} -\infty.
\]
Hence, for a sufficiently large $r > 0$ we have
\[
v(t, x) \leq \sup_{\mathbb{R}^n} g \quad \forall x \in \partial B_r(y) \text{ and } t \in [0, T].
\]
In view of the estimates (4.7) and (4.8) we obtain by the weak maximum principle on $(0, T] \times B_r(y)$:
\[
\max_{[0, T] \times B_r(y)} v \max_{(0) \times B_r(y)} v \leq \max_{\cup (0, T] \times \partial B_r(y)} v(t, x) \leq \sup g
\]
so that
\[
u(t, x) - \frac{\mu}{(T + \varepsilon - t)^{\gamma/2}} e^{\frac{|x-y|^2}{4(T + \varepsilon - t)}} = v(t, x) \leq \sup g \quad \forall t \in [0, T], \forall x \in \mathbb{R}^n
\]
and with $\mu \to 0$ we conclude
\[
u \leq \sup g \quad \text{on } \mathbb{R}_T^2, \quad \text{provided that } T < \frac{1}{4a}.
\]
If $T \geq \frac{1}{4a}$ decompose the time interval into
\[
[0, T] = [0, T_1] \cup \ldots \cup [T_k, T]
\]
with $T_{j+1} - T_j < \frac{1}{4a}$ and repeat the argument from above on each time interval $[T_j, T_{j+1}]$ (shift in time).
Corollary 4.17 (Uniqueness for the Cauchy problem). Let $T > 0$, $f \in C^0(\mathbb{R}_T^n; \mathbb{R})$, $g \in C^0(\mathbb{R}^n; \mathbb{R})$. Then there exists at most one solution $u \in C^2(\mathbb{R}_T^n; \mathbb{R}) \cap C^0(\overline{\mathbb{R}_T^n}; \mathbb{R})$ of the initial value problem

\[
\begin{align*}
  u_t - \Delta u &= f & \text{in } \mathbb{R}_T^n, \\
  u(0, b) &= g(b) & \text{for all } b \in \mathbb{R}^n,
\end{align*}
\]

which satisfies the growth condition

\[
|u(t, x)| \leq Ae^{a|x|^2} \quad \forall t \in [0, T] \quad \forall x \in \mathbb{R}^n.
\]  

(4.9)

for some positive constant $A, a > 0$.

Proof. Let $u, v$ be two solutions satisfying

\[
|u(t, x)| \leq A_1 e^{a_1|x|^2} \quad \text{and} \quad |v(t, x)| \leq A_2 e^{a_2|x|^2}.
\]

Then

\[
|u(t, x) - v(t, x)| \leq 2 \max\{A_1, A_2\} e^{\max\{a_1, a_2\}|x|^2}.
\]

The conclusion then follows from the maximum principle for the Cauchy problem (Theorem 4.16) applied to the functions $u - v$ and $v - u$.

Remark 4.18. Recall that for bounded $f$ and $g$ (and sufficient regularity) the solution of the Cauchy problem was given by (see Corollary 4.5):

\[
 u(t, x) = \int_{\mathbb{R}^n} \Phi(t, x - y)g(y) \, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(t - s, x - y)f(s, y) \, dy \, ds
\]

then, with $|g| \leq M$ and $|f| \leq M$ we obtain:

\[
|u(t, x)| \leq M \left( \int_{\mathbb{R}^n} \Phi(t, x - y) \, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(t - s, x - y)f(s, y) \, dy \, ds \right)_{t \leq T} \leq M (1 + T),
\]

meaning that this is the unique solution of the Cauchy problem which satisfies the growth condition (4.9).

Remark 4.19 (Non-uniqueness). The problem

\[
\begin{align*}
  u_t - \Delta u &= 0 & \text{in } \mathbb{R}_T^n, \\
  u(0, b) &= 0 & \text{for all } b \in \mathbb{R}^n,
\end{align*}
\]

has a trivial solution $u \equiv 0$, which, by Corollary 4.17, is the unique solution that satisfies a growth condition. However, there are infinitely many solutions which grow very rapidly as $|x| \to \infty$, see John: PDE, chapter 7.

Remark 4.20 (Regularity). If $u \in C^2(\Omega_T; \mathbb{R})$ solves the homogeneous heat equation in $\Omega_T$, then $u \in C^\infty(\Omega_T; \mathbb{R})$, for the proof we refer to Evans: PDE, Sec. 2.3 Thm. 8.

Remark 4.21. For interior estimates of derivatives of solutions of the homogeneous heat equation we also refer to Evans: PDE, Sec. 2.3 Thm. 9.

5 Excursion: Energy methods

5.1 Poisson equation

Theorem 5.1 (Uniqueness). Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $f \in C^0(\Omega; \mathbb{R})$ and $g \in C^0(\partial \Omega; \mathbb{R})$. There exists at most one solution $u \in C^2(\Omega; \mathbb{R}) \cap C^1(\overline{\Omega}; \mathbb{R})$ of the boundary value problem

\[
\begin{align*}
  -\Delta u &= f & \text{in } \Omega, \\
  u &= g & \text{on } \partial \Omega.
\end{align*}
\]
Proof. Let \( u, v \) be two solutions. Then \( w := u - v \) fulfills \( \Delta w \equiv 0 \) in \( \Omega \) and \( w \equiv 0 \) on \( \partial \Omega \). Hence:

\[
0 = \int_{\Omega} w \Delta w \, dx \quad \text{int. by parts} \quad \equiv 0 \quad - \int_{\Omega} \langle \nabla w, \nabla w \rangle \, dx + \int_{\partial \Omega} w \langle \nabla w, \nu \rangle \, dA(x) = - \int_{\Omega} |\nabla w|^2 \, dx,
\]

so that \( |\nabla w| \equiv 0 \) in \( \Omega \) \( \Rightarrow \) \( \nabla w \equiv 0 \) in \( \Omega \) \( \Rightarrow \) \( w \equiv \text{const in } \Omega \) \( \frac{w|_{\partial \Omega}}{w|_{\partial \Omega}} \equiv 0 \) \( \Rightarrow \) \( w \equiv 0 \) in \( \Omega \).

Consider the energy functional

\[
\mathcal{E}(v) := \int_{\Omega} \frac{1}{2} |\nabla v|^2 - vf \, dx
\]

and the admissible set of comparison functions

\[
\mathcal{C}(g) := \{ v \in C^2(\Omega; \mathbb{R}) \cap C^1(\overline{\Omega}; \mathbb{R}) : v = g \text{ on } \partial \Omega \}. \tag{5.2}
\]

**Theorem 5.2** (Dirichlet’s principle). Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain, \( f \in C^0(\Omega; \mathbb{R}) \), \( g \in C^0(\partial \Omega; \mathbb{R}) \) and \( u \in C^2(\Omega; \mathbb{R}) \cap C^1(\overline{\Omega}; \mathbb{R}) \) be a solution of the boundary value problem

\[
\left\{ \begin{array}{l}
-\Delta u = f & \text{in } \Omega, \\
u = g & \text{on } \partial \Omega.
\end{array} \right. \tag{BVP}
\]

Then

\[
\mathcal{E}(u) = \min_{v \in \mathcal{C}(g)} \mathcal{E}(v). \tag{5.1}
\]

Conversely, if \( u \in \mathcal{C}(g) \) satisfies (5.1), then \( u \) solves the boundary value problem (BVP).

Proof. “\( \Rightarrow \)” Let \( u \) be a solution of (BVP). For any \( v \in \mathcal{C}(g) \) we obtain:

\[
0 = \int_{\Omega} \langle -\Delta u, u - v \rangle \, dx \quad \text{int. by parts} \quad \equiv 0 \quad - \int_{\Omega} \langle \nabla u, \nabla (u - v) \rangle - (u - v) f \, dx.
\]

Hence, rearranging:\n
\[
\int_{\Omega} |\nabla u|^2 - uf \, dx = \int_{\Omega} \langle \nabla u, \nabla v \rangle - vf \, dx \leq \int_{\Omega} |\nabla u| |\nabla v| -vf \, dx \leq \int_{\Omega} \frac{1}{2} (|\nabla u|^2 + |\nabla v|^2) -vf \, dx,
\]

so that, we obtain

\[
\mathcal{E}(u) \leq \mathcal{E}(v) \quad \forall v \in \mathcal{C}(g).
\]

Since \( u \in \mathcal{C}(g) \), the minimum is attained and we conclude (5.1).

“\( \Leftarrow \)” Conversely, suppose that (5.1) is fulfilled. In particular we have

\[
\mathcal{E}(u) \leq \mathcal{E}(u + \varepsilon \varphi) \quad \forall |\varepsilon| < 1, \quad \forall \varphi \in C_c^\infty(\Omega; \mathbb{R}), \tag{5.2}
\]

since \( u + \varepsilon \varphi \in \mathcal{C}(g) \). Setting \( h(\varepsilon) := \mathcal{E}(u + \varepsilon \varphi) \) we have

\[
h(\varepsilon) = \int_{\Omega} \frac{1}{2} |\nabla u + \varepsilon \nabla \varphi|^2 - (u + \varepsilon \varphi) f \, dx = \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \varepsilon \langle \nabla u, \nabla \varphi \rangle + \frac{\varepsilon^2}{2} |\nabla \varphi|^2 - (u + \varepsilon \varphi) f \, dx.
\]

By (5.2) we have \( h(0) \leq h(\varepsilon) \) for all \( |\varepsilon| < 1 \), so that

\[
0 = h'(0) = \int_{\Omega} \langle \nabla u, \nabla \varphi \rangle - \varphi f \, dx \quad \text{int. by parts} \quad \equiv 0 \quad - \int_{\Omega} (-\Delta u - f) \varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega; \mathbb{R}),
\]

and by the fundamental lemma of the calculus of variations (Exercise 8) we conclude \( -\Delta u - f \equiv 0 \) in \( \Omega \). \( \square \)

\textit{In the second step we used the Cauchy–Bunyakovsky–Schwarz inequality for vectors} \( a, b \in \mathbb{R}^n \): \( \langle a, b \rangle \leq |a| |b| \).
5.2 Heat equation

**Theorem 5.3 (Uniqueness).** Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain, \( T > 0, f \in C^0(\Omega_T; \mathbb{R}) \) and \( g \in C^0(\Gamma_T; \mathbb{R}) \). There exists at most one solution \( u \in C^2(\Omega_T; \mathbb{R}) \cap C^1(\Omega^1_T; \mathbb{R}) \) of the problem

\[
\begin{align*}
    u_t - \Delta_x u &= f \quad \text{in } \Omega_T, \\
    u &= g \quad \text{on } \Gamma_T.
\end{align*}
\]

**Remark 5.4.** Attention: The index of the function classes refers here to the regularity with respect to the time variable.

**Proof.** Let \( u, v \) be two solutions. Then \( w := u - v \) satisfies

\[
\begin{align*}
w_t - \Delta_x w &= 0 \quad \text{in } \Omega_T, \\
w &= 0 \quad \text{on } \Gamma_T.
\end{align*}
\]

Consider

\[ h(t) := \int_\Omega [w(t, x)]^2 \, dx. \]

By the regularity assumptions we can differentiate with respect to the parameter \( t \) and obtain

\[
h'(t) = 2 \int_\Omega w w_t \, dx \overset{(5.3)}{=} 2 \int_\Omega \Delta_x w \, dx \overset{\text{int. by parts}}{=} -2 \int_\Omega \langle \nabla_x w, \nabla_x w \rangle \, dx = -2 \int_\Omega |\nabla_x w| \, dx \leq 0,
\]

meaning that \( h(\cdot) \) is monotone decreasing and we conclude for all \( t \in [0, T] \):

\[ 0 \leq h(t) \leq h(0) = \int_\Omega [w(0, x)]^2 \, dx \big|_{t=0} = 0. \]

Hence, \( h \equiv 0 \) and so \( w \equiv 0 \Rightarrow u = v \) in \( \Omega_T \).

**Theorem 5.5 (Backwards uniqueness).** Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain, \( T > 0, f \in C^0(\Omega_T; \mathbb{R}), g \in C^0([0, T] \times \partial \Omega; \mathbb{R}) \). and \( v, \bar{v} \in C^2(\Omega_T; \mathbb{R}) \) be two solutions of

\[
\begin{align*}
    u_t - \Delta_x u &= f \quad \text{in } \Omega_T, \\
    u &= g \quad \text{on } [0, T] \times \partial \Omega.
\end{align*}
\]

If \( v(T, b) = \bar{v}(T, b) \) for all \( b \in \Omega \), then \( v = \bar{v} \) in \( \overline{\Omega_T} \).

**Proof.** The function \( w := v - \bar{v} \) satisfies:

\[
\begin{align*}
w_t - \Delta_x w &= 0 \quad \text{in } \Omega_T, \\
w &= 0 \quad \text{on } [0, T] \times \partial \Omega.
\end{align*}
\]

Hence, we also have \( w_t \equiv 0 \) on \([0, T] \times \partial \Omega\). Again consider

\[ h(t) := \int_\Omega [w(t, x)]^2 \, dx \]

and as above we obtain

\[ h'(t) = -2 \int_\Omega |\nabla_x w|^2 \, dx. \]

Since

\[
\frac{\partial}{\partial t} |\nabla_x w|^2 = \frac{\partial}{\partial t} \langle \nabla_x w, \nabla_x w \rangle = \langle \frac{\partial}{\partial t} \nabla_x w, \nabla_x w \rangle + \langle \nabla_x w, \frac{\partial}{\partial t} \nabla_x w \rangle = 2 \langle \nabla_x w_t, \nabla_x w \rangle
\]

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differentiating again it follows
\[ h''(t) = -4 \int_\Omega \langle \nabla_x w_t, \nabla_x w \rangle \, dx \]
int. by parts
\[ w_t \bigg|_{t=0} = 0 \quad \forall t \in [0,T] \]
and
\[ 4 \int_\Omega w_t \Delta_x w \, dx \]
by (5.4) we obtain
\[ \Delta_x u \equiv 0 \quad \forall \Omega \]
Moreover, we have
\[ \int_\Omega |\nabla_x w|^2 \, dx = \int_\Omega \langle \nabla_x w, \nabla_x w \rangle \, dx \]
int. by parts
\[ w_t \bigg|_{t=0} = 0 \quad \forall t \in [0,T] \]
by H"older ineq.
\[ \left( \int w^2 \, dx \right)^{1/2} \leq \left( \int (\Delta_x w)^2 \, dx \right)^{1/2} \]
Hence, by (5.4) we obtain
\[ [h'(t)]^2 = 4 \left[ \int_\Omega |\nabla_x w|^2 \, dx \right] \leq h(t) h''(t) \quad \forall t \in [0, T]. \] (5.5)
Now, if \( h \equiv 0 \Rightarrow w \equiv 0 \Rightarrow v = \overline{v} \) in \( \Omega_T \). Otherwise, since from the assumptions we have \( h(T) = 0 \), moreover \( h \geq 0 \) and is monotone decreasing, there would exist \( 0 \leq t_1 < t_2 \leq T \) such that
\[ h(t) > 0 \quad \forall t \in [t_1, t_2) \quad \text{and} \quad h(t_2) = 0. \]
Consider
\[ h(t) := \ln h(t) \quad \text{for} \ t \in [t_1, t_2). \]
Then:
\[ h''(t) = \frac{d}{dt} \left[ \frac{h'(t)}{h(t)} \right] = \frac{h''(t) h(t) - [h'(t)]^2}{[h(t)]^2} \geq 0 \] (5.5)
meaning that \( h(\cdot) \) is convex in the interval \([t_1, t_2)\). Thus:
\[ h((1 - \lambda) t_1 + \lambda t_2) \leq (1 - \lambda) h(t_1) + \lambda h(t_2) \quad \forall \lambda \in (0, 1) \quad \forall t \in (t_1, t_2) \]
By continuity of \( h(\cdot) \) we conclude for \( t \to t_2 \):
\[ 0 \leq h((1 - \lambda) t_1 + \lambda t_2) \leq [h(t_1)]^{1-\lambda} [h(t_2)]^\lambda = 0 \quad \forall \lambda \in (0, 1), \]
so that we have \( h(t) = 0 \) for all \( t \in [t_1, t_2] \) in contrast to the choice of \( t_1 \).

5.3 Wave equation
\[ u_{tt} - \Delta_x u \equiv 0 \quad \text{in} \ \Omega_T \quad \Rightarrow \quad \text{vibrating string} \]

**Theorem 5.6** (Uniqueness). Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain, \( T > 0 \), \( f \in C^0(\Omega_T; \mathbb{R}) \), \( g \in C^0(\Gamma_T; \mathbb{R}) \) and \( h \in C^0(\Omega; \mathbb{R}) \). There exists at most one solution \( u \in C^2(\Omega_T; \mathbb{R}) \) of
\[
\begin{cases}
  u_{tt} - \Delta_x u = f & \text{in} \ \Omega_T, \\
  u = g & \text{on} \ \Gamma_T, \\
  u_t(0,b) = h(b) & \forall \ b \in \Omega.
\end{cases}
\]
Proof. Let $u, v$ be two solutions. Then $w := u - v$ satisfies

$$\begin{cases}
w_t - \Delta_x w \equiv 0 & \text{in } \Omega_T, \\
w \equiv 0 & \text{on } \Gamma_T, \\
w_t(0, b) \equiv 0 & \forall \ b \in \Omega.
\end{cases}$$

Consider for $t \in [0, T]$:

$$h(t) := \frac{1}{2} \int_{\Omega} |\nabla_{(t,x)} w(t, x)|^2 \, dx = \frac{1}{2} \int_{\Omega} |w_t(t, x)|^2 + |\nabla_x w(t, x)|^2 \, dx.$$ 

Then:

$$h'(t) = \int_{\Omega} w_t w_{tt} + \langle \nabla_x w, \nabla_x w_t \rangle \, dx \quad \text{int. by parts w/ resp. } \nabla_x w = \int_{\Omega} (w_{tt} - \Delta_x w) w_t \, dx \equiv 0,$$

and the boundary term vanishes since:

$$w \equiv 0 \quad \text{on } [0, T] \times \partial \Omega \quad \Rightarrow \quad w_t \equiv 0 \quad \text{on } [0, T] \times \partial \Omega.$$

Hence: $h(\cdot) \equiv \text{const}$. Since $w_t(0, b) \equiv 0$ and $w(0, b) \equiv 0 \Rightarrow \nabla_x w(0, b) \equiv 0$ we obtain $h(t) = h(0) = 0$ 

$\Rightarrow \nabla_{(t,x)} w \equiv 0 \quad \Rightarrow \quad w \equiv \text{const} \quad \Rightarrow \quad w \equiv 0 \quad \text{in } \Omega_T.$

\[\square\]

Cavalieri’s principle

Recall that by Cavalieri’s principle we have

$$\int_{B_r(a)} f(x) \, dx = \int_0^r \int_{\partial B_\rho(a)} f(x) \, dA(x) \, d\rho$$

$$\Rightarrow \quad \frac{d}{dr} \left( \int_{B_r(a)} f(x) \, dx \right) = \int_{\partial B_r(a)} f(x) \, dA(x) \quad (5.6)$$

Theorem 5.7 (Finite propagation speed). Let $u \in C^2([0, \infty) \times \mathbb{R}^n; \mathbb{R})$ be a solution of $u_{tt} - \Delta_x u \equiv 0$ in $(0, \infty) \times \mathbb{R}^n$. If $u(0, b) = u_t(0, b) = 0$ for all $b \in B_\vartheta(a)$, then

$$u \equiv 0 \quad \text{in the cone } C := \{ (t, x) \in [0, \infty) \times \mathbb{R}^n : t \in [0, \vartheta), |x - a| \leq \vartheta - t \}.$$
Proof. Consider
\[ h(t) := \frac{1}{2} \int_{B_{\theta^{-1}}(a)} |\nabla_{(t,x)} u|^2 \, dx = \frac{1}{2} \int_{\partial B_{\theta^{-1}}(a)} u_t^2 + |\nabla_x u|^2 \, dx. \]
Differentiating with respect to \( t \), we obtain in view of (5.6):
\[
h'(t) = \int_{B_{\theta^{-1}}(a)} u_t u_{tt} + \langle \nabla_x u, \nabla_x u_t \rangle \, dx + \frac{1}{2} \int_{\partial B_{\theta^{-1}}(a)} u_t^2 + |\nabla_x u|^2 \, dA(x) \tag{-1}
\]
\[ \text{int. by parts} \]
\[ w/ \text{resp.} \ x \]
\[ \text{w} \text{CBS} \]
\[ |\nu|=1 \int_{\partial B_{\theta^{-1}}(a)} |u_t| |\nabla_x u| - \frac{1}{2} u_{tt} - \frac{1}{2} |\nabla_x u|^2 \, dA(x) \leq 0 \]
meaning that \( h(\cdot) \) is monotone decreasing.
Thus, \( 0 \leq h(t) \leq h(0) = 0 \) since \( u_t(0, b) = 0 \) and \( u(0, b) = 0 \) \( \Rightarrow \nabla_x u(0, b) = 0 \).
\[ \Rightarrow h(t) \equiv 0 \Rightarrow \nabla_{(t,x)} u \equiv 0 \Rightarrow u \equiv \text{const.} \]
\[ \Rightarrow u \equiv 0 \text{ in the cone } C. \]

6 Wave equation

6.1 Physical interpretation
Consider the hyperbolic PDE
\[ u_{tt} - \Delta_x u \equiv 0 \tag{6.1} \]
for all \( t \in I \subseteq \mathbb{R} \) and all \( x \in \Omega \text{(domain)} \subseteq \mathbb{R}^n \) with \( u : I \times \Omega \to \mathbb{R}, (t, x) \mapsto u(t, x) \).

The homogeneous PDE (6.1) is a simplified model for
- waves in acoustics or optics \((n = 3)\),
- waves on membranes or water \((n = 2)\),
- sound waves in pipes or vibrating strings \((n = 1)\).

Hereby the function \( u \) represents the displacement \( u \) is the velocity and \( u_{tt} \) the acceleration.
Hence, \( u_{tt} - \Delta_x u \equiv 0 \) "says" the acceleration of the displacement is proportional to the changes in a neighborhood which are "described" by \( \Delta_x u \).

in Physics: presence of wave speed \( c > 0 \):
\[ u_{tt} - c^2 \Delta_x u \equiv 0 \]
\( \Rightarrow \) change time unit (normalize the equation to \( c = 1 \)) and consider \( v(t, x) := u \left( \frac{t}{c}, x \right) \).
Then
\[ v_{tt}(t, x) = \frac{\partial}{\partial t} \left[ \frac{1}{c} u_t \left( \frac{t}{c}, x \right) \right] = \frac{1}{c^2} u_{tt} \left( \frac{t}{c}, x \right) = \Delta_x u \left( \frac{t}{c}, x \right) = \Delta_x v(t, x) \]
satisfies the homogeneous wave equation with \( c = 1 \). Thus, w.l.o.g. we assume \( c = 1 \) in the following.

6.2 \( n = 1 \)
Let \( u \in C^2 \) satisfy \( \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} \equiv 0 \) for \( x \in \mathbb{R}, t \geq 0 \). Introducing the so-called characteristic coordinates
\[ \xi = x + t \quad \text{and} \quad \eta = x - t \]
we obtain
\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial t} \right) \quad \text{chain rule} \quad \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial \xi} \frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \eta} \frac{\partial u}{\partial \xi} \right) = \frac{\partial^2 u}{\partial \xi^2} \frac{\partial}{\partial \xi} + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial}{\partial \eta}
\]
\[
= \frac{\partial^2 u}{\partial \xi^2} - 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}
\]
as well as
\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) \quad \text{chain rule} \quad \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial \xi} \frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \eta} \frac{\partial u}{\partial \xi} \right) = \frac{\partial^2 u}{\partial \xi^2} \frac{\partial}{\partial \xi} \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial}{\partial \eta} \frac{\partial u}{\partial x}
\]
\[
= \frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \xi^2} \frac{\partial u}{\partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial u}{\partial \xi} + \frac{\partial^2 u}{\partial \xi \partial \eta} \frac{\partial^2 u}{\partial \eta \partial \xi}
\]
so that the wave equation in characteristic coordinates reads
\[
0 \equiv \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} - 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} - \left( \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \xi \partial \eta} \right)
\]
and simplifies to
\[
\frac{\partial^2 u}{\partial \xi \partial \eta} = 0
\]
meaning that \( \frac{\partial u}{\partial \eta} \) is independent of \( \xi \) and can be written as \( \frac{\partial u}{\partial \eta} = \tilde{w}_1(\eta) \), and integrating we obtain
\[
u(\xi, \eta) = \int \tilde{w}_1(\eta) \, d\eta + w_2(\xi) = w_1(\eta) + w_2(\xi)
\]
for some functions \( w_1, w_2 \in C^2(\mathbb{R}; \mathbb{R}) \). Hence, in the original coordinates we have
\[
u(t, x) = w_1(x - t) + w_2(x + t).
\]
Conversely, for arbitrary \( w_1, w_2 \in C^2(\mathbb{R}; \mathbb{R}) \) the function \( \nu(t, x) := w_1(x - t) + w_2(x + t) \) is a \( C^2 \)-solution of the homogeneous wave equation \( u_{tt} - u_{xx} \equiv 0 \).

**Example:** \( w_2 \equiv 0 \), i.e., \( \nu(t, x) := w_1(x - t) \) and increasing \( x \) and \( t \) by the same value does not change the value of the displacement \( \nu \), so that we obtain a "travelling wave".

![Figure 16: A travelling wave, indicated at three time steps.](image)

Now we add initial conditions at time \( t = 0 \):
\[
\begin{align*}
\{ & u(0, x) = g(x), \\
& u_t(0, x) = h(x), \quad \forall x \in \mathbb{R},
\end{align*}
\]
with the prescribed functions \( g \in C^2(\mathbb{R}; \mathbb{R}) \) and \( h \in C^1(\mathbb{R}; \mathbb{R}) \). Then
\[
u(0, x) = w_1(x) + w_2(x) = g(x),
\]
and
\[
u_t(0, x) = \frac{\partial}{\partial t} (w_1(x - t) + w_2(x + t)) \bigg|_{t=0} = (-w_1'(x - t) + w_2'(x + t)) \bigg|_{t=0} = -w_1'(x) + w_2'(x) = h(x).
\]
Integrating the second equation:
\[
-w_1(x) + w_2(x) = \frac{1}{2} \int_0^x h(y) \, dy + c.
\]
Adding and subtracting (6.2) and (6.3) we deduce
\[
w_2(x) = \frac{1}{2} \left[ g(x) + \int_0^x h(y) \, dy + c \right] \quad \text{and} \quad w_2(x) = \frac{1}{2} \left[ g(x) - \int_0^x h(y) \, dy - c \right],
\]
so that we conclude
\[
u(t, x) = w_1(x - t) + w_2(x + t) = \frac{1}{2} \left[ g(x - t) + g(x + t) + \int_{x-t}^{x+t} h(y) \, dy \right].
\]
This is d'Alembert's formula and represents the unique solution of the initial value problem of the wave equation in 1D.

Indeed, we have derived (6.4) assuming \( u \) is a \( C^2 \)-solution, also the converse holds true:

**Theorem 6.1** (Solution of the wave equation in 1D). Let \( g \in C^2(\mathbb{R}; \mathbb{R}) \) and \( h \in C^1(\mathbb{R}; \mathbb{R}) \). Then, \( u \) defined by (6.4) satisfies
\begin{enumerate}[i)]
\item \( u \in C^2([0, \infty) \times \mathbb{R}; \mathbb{R}) \),
\item \( u_{tt}(t, x) - u_{xx}(t, x) \equiv 0 \) for all \( t \geq 0, x \in \mathbb{R} \),
\item \( \lim_{t \to 0^+, x \to 0} u_{tt}(t, x) = g(0) \) and \( \lim_{t \to 0^+, x \to 0} u_t(t, x) = h(0) \) for all \( b \in \mathbb{R} \).
\end{enumerate}

**Proof.** Direct calculation, which is left as an exercise to the reader. \( \square \)

**Remark 6.2.**
\begin{itemize}
\item We see from d'Alembert's formula (6.4) that if \( g \in C^k \) and \( h \in C^{k-1} \), then \( u \in C^k \), but is not in general smoother!
Hence, the wave equation does not cause smoothing.
\item D'Alembert's formula also shows that the value of \( u(\vartheta, a) \) is uniquely determined by the values of the initial functions in the interval \([a - \vartheta, a + \vartheta]\) of the \( x \)-axis, this interval is called domain of dependence for the solution at point \((\vartheta, a)\).
The values of \( g \) and \( h \) inside \([a - \vartheta, a + \vartheta]\) determine uniquely the solution in the cone, which is bounded by the characteristics through \((\vartheta, a)\), cf. Theorem 5.7 and Fig 17.
Conversely, the initial values at a point \((0, b)\) of the \( x \)-axis influence \( u(t, x) \) in the region bounded by the characteristics through \((0, b)\), see Fig. 18.
\end{itemize}
So far: solved Cauchy problem for the 1D wave equation (i.e. initial data is given on the whole space (here: \(\mathbb{R}\)) and not boundary conditions). Now, consider the 1D wave equation on a bounded interval \([0, l]\) with additional boundary conditions

\[
\begin{aligned}
&u(t, 0) = u(t, l) = 0 \quad \forall \ t \geq 0,
\end{aligned}
\]

(e.g. vibrating string fixed at endpoints). So, let \(u \in C^2([0, \infty) \times [0, l]; \mathbb{R})\) be a solution of

\[
\begin{cases}
    u_{tt} - u_{xx} = 0 & \text{in } [0, \infty) \times [0, l], \\
    u(t, 0) = u(t, l) = 0 & \text{for all } t \geq 0 \quad \text{boundary values}, \\
    u(0, b) = g(b), \quad u_t(0, b) = h(b) & \text{for all } b \in [0, l] \quad \text{initial values},
\end{cases}
\]

with \(g \in C^2([0, \infty) \times [0, l]; \mathbb{R})\) and \(h \in C^1([0, l]; \mathbb{R})\).

\[
\begin{align*}
\text{boundary values} & \implies \quad \frac{\partial u}{\partial t}(t, 0) \equiv 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2}(t, 0) \equiv 0 \quad \forall \ t \geq 0 \\
\text{wave eq.} & \implies \quad \frac{\partial^2 u}{\partial x^2}(t, 0) \equiv 0 \quad \forall \ t \geq 0.
\end{align*}
\]

On the other hand by the initial values we have for all \(x \in [0, l]\):

\[
\begin{aligned}
    u(0, x) = g(x) \implies \quad \frac{\partial^2 u}{\partial x^2}(0, x) = g''(x)
\end{aligned}
\]

and in particular

\[
\begin{align*}
0 & \overset{\text{BV}}{=} u(0, 0) \overset{\text{IV}}{=} g(0) \quad \text{and} \quad 0 \overset{\text{(6.5)}}{=} \frac{\partial^2 u}{\partial x^2}(0, 0) \overset{\text{(6.6)}}{=} g''(0),
\end{align*}
\]

meaning that as necessary condition we gain:

\[
\begin{align*}
g(0) = 0 \quad \text{and} \quad g''(0) = 0.
\end{align*}
\]

Similar we conclude the validity:

\[
\begin{align*}
g(l) = 0 \quad \text{and} \quad g''(l) = 0.
\end{align*}
\]

Moreover, we obtain with the initial values:

\[
\begin{align*}
0 \overset{\text{(6.5)}}{=} u_t(0, 0) \overset{\text{IV}}{=} h(0) \quad \text{as well as} \quad h(l) = 0.
\end{align*}
\]

Hence the compatibility conditions (which are necessary for the existence of solutions) read:

\[
\begin{align*}
g(0) = g(l) = 0, \quad g''(0) = g''(l) = 0, \quad h(0) = h(l) = 0.
\end{align*}
\]

From now on, assume that these compatibility conditions are fulfilled. Idea to derive a formula for the solution: use reflection method to extend the initial values to the whole space (here: \(\mathbb{R}\)), then apply d’Alembert’s formula:

**Start with the odd reflection** (i.e. point reflection across zero) and extend the initial data to \([-l, l]\):

\[
\begin{align*}
\hat{g}(x) := \begin{cases}
    g(x), & x \in [0, l] \\
    -g(-x), & x \in [-l, 0]
\end{cases}
\quad \text{and} \quad
\hat{h}(x) := \begin{cases}
    h(x), & x \in [0, l] \\
    -h(-x), & x \in [-l, 0]
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\hat{g} \in C^2([-l, l]; \mathbb{R}), \quad \hat{h} \in C^1([-l, l]; \mathbb{R}).
\end{align*}
\]

![Figure 19: Indicated is an odd reflection together with a 2l-periodic extension.](image-url)
Next: 2l-periodic extension to the whole $\mathbb{R}$ (see Fig. 19):

$$\tilde{g}(x + 2l) := \tilde{g}(x) \quad \text{and} \quad \tilde{h}(x + 2l) := \tilde{h}(x) \quad \forall x \in [-l, l], \forall k \in \mathbb{Z} \implies \tilde{g} \in C^2(\mathbb{R}; \mathbb{R}), \quad \tilde{h} \in C^1(\mathbb{R}; \mathbb{R})$$

and the data is extended to the whole space, we have a Cauchy problem and apply d’Alembert’s formula to obtain its unique solution

$$\tilde{u}(t, x) := \frac{1}{2} \left[ \tilde{g}(x + t) + \tilde{g}(x - t) + \int_{x-t}^{x+t} \tilde{h}(y) \, dy \right]$$

So that by Theorem 6.1 we conclude

- $\tilde{u} \in C^2([0, \infty) \times \mathbb{R}; \mathbb{R})$ and in particular $u := \tilde{u}|_{[0, \infty) \times [0, l]} \in C^2([0, \infty) \times [0, l]; \mathbb{R})$,
- $\tilde{u}_{tt} - \tilde{u}_{xx} \equiv 0$ on $[0, \infty) \times \mathbb{R}$ and in particular $u_{tt} - u_{xx} \equiv 0$ on $[0, \infty) \times [0, l]$,
- $\tilde{u}(0, b) = \tilde{g}(b), \tilde{u}_t(0, b) = \tilde{h}(b)$ for all $b \in \mathbb{R}$ and in particular $u(0, b) = g(b), u_t(0, b) = h(b)$ for all $b \in [0, l]$,

and it only remains to check the boundary conditions:

$$u(t, 0) = \tilde{u}(t, 0) = \frac{1}{2} \left[ \tilde{g}(t) + \tilde{g}(-t) + \int_{-t}^{t} \tilde{h}(y) \, dy \right] = \frac{1}{2} \left[ \tilde{g}(t) - \tilde{g}(t) + \int_{0}^{t} \tilde{h}(y) \, dy - \int_{-t}^{0} \tilde{h}(y) \, dy \right] = 0,$$

as well as

$$u(t, l) = \tilde{u}(t, l) = \frac{1}{2} \left[ \tilde{g}(l + t) + \tilde{g}(l - t) + \int_{l-t}^{l+t} \tilde{h}(y) \, dy \right] = \frac{1}{2} \left[ \tilde{g}(l + t) + \tilde{g}(-l - t) + \int_{l}^{l+t} \tilde{h}(y) \, dy + \int_{-(l+t)}^{-l} \tilde{h}(y) \, dy \right] = 0.$$

### 6.3 Spherical means

Let $v \in C^2(\mathbb{R}^n; \mathbb{R})$. Its spherical mean

$$(Mv)(a, r) := \int_{\partial B_r(a)} v(y) \, dA(y) = \frac{1}{nV_n} \int_{\partial B_r(a)} v(y) \, dA(y) = \left\{ \begin{array}{ll} y = a + rz \Rightarrow & \{ y = a + rz \} \\
A(y) = r^{n-1} \, dA(z) \end{array} \right.$$  \hfill (6.7)

hence, differentiating with respect to the radius we obtain:

$$\frac{\partial}{\partial r} (Mv)(a, r) \overset{(6.7)}{=} \frac{1}{nV_n} \frac{\partial}{\partial B_r(a)} \int_{\partial B_r(a)} v(a + rz) \, dA(z) \overset{\text{comp.}}{=} \frac{1}{nV_n} \int_{\partial B_r(a)} \frac{\partial}{\partial r} (v(a + rz)) \, dA(z)$$

$$= \frac{1}{nV_n} \int_{\partial B_r(a)} (D_xv)(a + rz) \cdot z \, dA(z) \overset{z = rz}{=} \frac{1}{nV_n} \int_{\partial B_r(a)} \langle (\nabla_x v)(a + rz), \nu(z) \rangle \, dA(z)$$

$$\overset{\text{div. thm.}}{=} \frac{1}{nV_n} \int_{B_r(0)} \text{div}_z \left( (\nabla_x v)(a + rz) \right) \, dz = \frac{1}{nV_n} \int_{B_r(0)} r \cdot \left( \text{div}_x \nabla_x v)(a + rz) \right) \, dz$$

$$= \frac{r}{nV_n} \int_{B_r(0)} (\Delta_x v)(a + rz) \, dz \quad \text{(6.8)}$$

Let’s pause for a moment and see that with the previous calculations we obtain a new proof of Corollary 3.6, i.e. we show that if $v \in C^2(\mathbb{R}^n; \mathbb{R})$ is harmonic ($\Delta v \equiv 0$), then $u$ has the mean value property. Indeed, it follows form (6.8) that for all $a \in \mathbb{R}^n$:

$$\frac{\partial}{\partial r} (Mv)(a, r) \equiv 0 \quad \Rightarrow \quad (Mv)(a, r) \equiv \text{const} \quad \forall r > 0.$$
Hence, sending $r \downarrow 0$ in (6.7) we conclude:

$$\begin{align*}
(Mv)(a,r)^{(6.7)} &= \frac{1}{nV_n} \int_{\partial B_1(0)} v(a + rz) \, dA(z) = \lim_{r \downarrow 0} \frac{1}{nV_n} \int_{\partial B_1(0)} v(a + rz) \, dA(z) \\
\text{and } v \in C^0 &\quad \frac{1}{nV_n} \int_{\partial B_1(0)} \lim_{r \downarrow 0} v(a + rz) \, dA(z) = \frac{1}{nV_n} \int_{\partial B_1(0)} v(a) \, dA(z) = v(a).
\end{align*}$$

Let’s now continue with the $r$-derivative of the spherical mean:

$$\begin{align*}
\frac{\partial}{\partial r}(Mv)(a,r)^{(6.8)} &= \frac{r}{nV_n} \int_{B_1(0)} (\Delta_x v)(a + rz) \, dz = \frac{r}{nV_n} \int_{B_1(0)} \Delta_a v(a + rz) \, dz \\
\text{and } v \in C^2 &\quad \frac{r}{nV_n} \Delta_a \int_{B_1(0)} v(a + rz) \, dz = \left\{ x = a + rz \ \text{and } \ dx = r^n \, dz \right\} = \frac{1}{nV_n r^{n-1}} \Delta_a \int_{B_r(a)} v(x) \, dx \\
\text{Cavalieri} &\quad \frac{1}{nV_n r^{n-1}} \Delta_a \int_0^r \int_{\partial B_r(a)} v(x) \, dA(x) \, d\rho = \frac{1}{r^{n-1}} \Delta_a \int_0^r \int_{\partial B_r(a)} v(x) \, dA(x) \, d\rho
\end{align*}$$

Hence:

$$r^{n-1} \frac{\partial}{\partial r}(Mv)(a,r) = \Delta_a \int_0^r \rho^{n-1}(Mv)(a,\rho) \, d\rho$$

and differentiating both sides with respect to $r$ we obtain

$$(n - 1)r^{n-2} \frac{\partial}{\partial r}(Mv)(a,r) + r^{n-1} \frac{\partial^2}{\partial r^2}(Mv)(a,r) = \Delta_a \left( r^{n-1}(Mv)(a,r) \right),$$

$$\Rightarrow \frac{\partial^2}{\partial r^2}(Mv)(x,r) + \frac{n - 1}{r} \frac{\partial}{\partial r}(Mv)(x,r) = \Delta_x (Mv)(x,r).$$

This is **Darboux’s equation**, which is valid for all $v \in C^2(\mathbb{R}^n; \mathbb{R})$.

Let now $u \in C^2(0, \infty) \times \mathbb{R}^n; \mathbb{R}$ denote a solution of the homogeneous wave equation $u_{tt} - \Delta_x u \equiv 0$ in $[0, \infty) \times \mathbb{R}^n$. Then, for fixed time $t$, the spherical mean $(Mu(t, \cdot))(x,r)$ satisfies Darboux’s equation:

$$\frac{\partial^2}{\partial r^2}(Mu(t,\cdot))(x,r) + \frac{n - 1}{r} \frac{\partial}{\partial r}(Mu(t,\cdot))(x,r) = \Delta_x (Mu(t,\cdot))(x,r) = \frac{1}{nV_n} \Delta_x \int_{\partial B_1(0)} u(t,x + rz) \, dA(z)$$

$$(u(t,\cdot) \in C^2) \quad \frac{1}{nV_n} \int_{\partial B_1(0)} \Delta_x u(t,x + rz) \, dA(z) = \frac{1}{nV_n} \int_{\partial B_1(0)} \frac{\partial^2 u}{\partial t^2}(t,x + rz) \, dA(z)$$

$$(u \in C^2) \quad \frac{\partial^2}{\partial r^2} \left( \frac{1}{nV_n} \int_{\partial B_1(0)} u(t,x + rz) \, dA(z) \right) = \frac{\partial^2}{\partial t^2} (Mu(t,\cdot))(x,r).$$

Summarizing, $(Mu(t,\cdot))(x,r)$ as a function of the two scalar variables $t$, $r$ for fixed $x$ is a solution of

$$\frac{\partial^2}{\partial t^2} (Mu(t,\cdot))(x,r) = \frac{\partial^2}{\partial r^2} (Mu(t,\cdot))(x,r) + \frac{n - 1}{r} \frac{\partial}{\partial r} (Mu(t,\cdot))(x,r).$$

(EPD)

This PDE depends on the spatial dimension $n$ and is called **Euler-Poisson-Darboux** equation.
Remark 6.3. The differential operator on the right-hand side $\frac{\partial^2}{\partial t^2} + \frac{n-1}{r} \frac{\partial}{\partial r}$ is the radial part of the Laplacian in polar coordinates (cf. Exercise 12). Hence, if $u$ is a radial symmetric solution of the homogeneous wave equation, we conclude with (EPD) that also the spherical means of $u$ are solutions of the homogeneous wave equation. This is expected, since the wave equation is invariant under spatial rotations, cf. Exercise 32 b).

The remaining question is: How to solve the Euler-Poisson-Darboux equation? \[\rightarrow\] In general: non-trivial!

6.4 $n = 3$

Lemma 6.4. Let $u \in C^2([0, \infty) \times \mathbb{R}^3; \mathbb{R})$ be a solution of the Cauchy problem

\[
\begin{cases}
  u_{tt} - \Delta_x u \equiv 0, & \text{in } [0, \infty) \times \mathbb{R}^3, \\
  u(0, x) = g(x), \ u_t(0, x) = h(x), & \text{for all } x \in \mathbb{R}^3,
\end{cases}
\]

with $g \in C^2(\mathbb{R}^3; \mathbb{R})$ and $h \in C^1(\mathbb{R}^3; \mathbb{R})$. Then the function

$$w(x, r, t) := r \cdot (Mu(t, \cdot))(x, r)$$

is $C^2$ for all $x \in \mathbb{R}^3, t \in [0, \infty), r \in [0, \infty)$ and satisfies

\[
\begin{cases}
  w_{tt} - w_{rr} \equiv 0, & \forall x \in \mathbb{R}^3, \forall t \geq 0, \forall r \geq 0, \\
  w(x, r, 0) = r \cdot (Mg)(x, r), \ w_t(x, r, 0) = r \cdot (Mh)(x, r), & \forall x \in \mathbb{R}^3, \forall r \geq 0.
\end{cases}
\]

Proof. We have

$$w_{tt} = r \cdot \frac{\partial^2}{\partial t^2} (Mu(t, \cdot))(x, r) \quad \text{(EPD)} = r \cdot \frac{\partial^2}{\partial t^2} (Mu(t, \cdot))(x, r) + 2 \cdot \frac{\partial}{\partial r} (Mu(t, \cdot))(x, r)$$

$$= \frac{\partial}{\partial r} \left[ (Mu(t, \cdot))(x, r) + r \cdot \frac{\partial}{\partial r} (Mu(t, \cdot))(x, r) \right] = \frac{\partial^2}{\partial r^2} [r \cdot (Mu(t, \cdot))(x, r)] = w_{rr}.$$

Moreover:

$$w(x, r, 0) = r \cdot (Mu(0, \cdot))(x, r) = r \cdot (Mg)(x, r)$$

and

$$w_t(x, r, 0) = r \cdot \frac{\partial}{\partial t} (Mu(t, \cdot))(x, r) \bigg|_{t=0} = r \cdot \left( M \frac{\partial}{\partial t} u(t, \cdot) \right) (x, r) \bigg|_{t=0} = r \cdot (Mu_t(0, \cdot))(x, r) = r \cdot (Mh)(x, r).$$

Situation in 0: First, note that

$$\lim_{r \searrow 0} (Mu(t, \cdot))(x, r) \xrightarrow{n=3} \lim_{r \searrow 0} \frac{1}{3} \int_{\partial B_1(0)} u(t, x + rz) \, dA(z) \quad u \in C^0 \quad \frac{1}{4\pi} \int_{\partial B_1(0)} u(t, x) \, dA(z) = u(t, x) \quad (6.9)$$

Hence, $w(x, r, t) = r \cdot (Mu(t, \cdot))(x, r) \xrightarrow{r \searrow 0} 0$ and also all desired derivatives exists in the endpoints for $r \searrow 0$ as well as for $t \searrow 0$ (see also the proof of the next theorem).

Theorem 6.5 (Kirchhoff’s formula). The Cauchy problem for the wave equation in 3D

\[
\begin{cases}
  u_{tt} - \Delta_x u \equiv 0, & \text{in } [0, \infty) \times \mathbb{R}^3, \\
  u(0, x) = g(x), \ u_t(0, x) = h(x), & \text{for all } x \in \mathbb{R}^3,
\end{cases}
\]

with $g \in C^3(\mathbb{R}^3; \mathbb{R})$ and $h \in C^2(\mathbb{R}^3; \mathbb{R})$ has a unique solution $u \in C^2([0, \infty) \times \mathbb{R}^3; \mathbb{R})$, this solution is given by Kirchhoff’s formula

$$u(t, x) = \frac{1}{4\pi t^2} \int_{\partial B_t(x)} g(z) + \langle \nabla g(z), z - x \rangle + th(z) \, dA(z).$$
Remark 6.6.  • In contrast to d’Alembert’s formula (sol. of wave equation in 1D), Kirchhoff’s formula for the solution of the wave equation in 3D involves the derivatives of $g$, meaning that the solution in 3D may even be less regular that the initial data ($g \in C^3 \not\sim u \in C^2$).

• The domains of dependence in 3D are spheres, see Fig. 20, more precisely: The value of $u(\vartheta, a)$ is uniquely determined by the values of the initial data and their derivatives on the sphere $\partial B_\vartheta(a) \subset \mathbb{R}^3$ (space time at $t = 0$).

Figure 20: Domains of dependence in 3D are spheres.

Proof of Theorem 6.5. The proof consists of two steps:

1. Show that any solution has the asserted form $\sim$ uniqueness.

2. Show that the function defined by Kirchhoff’s formula is indeed a solution $\sim$ existence.

ad 1. Let $u$ be a $C^2$-solution of the Cauchy problem. Then by Lemma 6.4 the function

$$w(x, r, t) := r \cdot (Mu(t, \cdot))(x, r)$$

is a $C^2$-solution of the wave equation in 1D on the half-line $\{ r \geq 0 \}$:

$$\begin{align*}
   &w_{tt} - w_{rr} \equiv 0, \\
   &w(x, r, 0) = r \cdot (Mg)(x, r), \quad w_t(x, r, 0) = r \cdot (Mh)(x, r),
\end{align*}$$

∀ $x \in \mathbb{R}^3, \forall t \geq 0, \forall r \geq 0$,

We extend $w$ by an odd reflection to $(-\infty, 0)$:

$$\tilde{w}(x, r, t) := \begin{cases} w(x, r, t), & r \geq 0, \\
-w(x, -r, t), & r < 0. \end{cases}$$

Note that $\tilde{w}(x, 0, t) = w(x, 0, t) = 0$ by the definition of $w$ and for $r > 0$:

$$\frac{\partial \tilde{w}}{\partial r}(x, r, t) = \frac{\partial w}{\partial r}(x, r, t) \prod \text{rule} = (Mu(t, \cdot))(x, r) + r \cdot \frac{\partial}{\partial r}(Mu(t, \cdot))(x, r)$$

$$= \frac{1}{4\pi} \int_{\partial B_1(0)} u(t, x + rz) \, dA(z) + \frac{r}{4\pi} \int_{\partial B_1(0)} (\nabla_z u(t, x + rz), z) \, dA(z)$$

and the limit exists as $r \searrow 0$. Then

$$\lim_{r \searrow 0} \left. \frac{\tilde{w}(x, r, t) - \tilde{w}(x, 0, t)}{r} \right|_{r=0} = \lim_{r \searrow 0} \left. \frac{w(x, -r, t) - w(x, 0, t)}{-r} \right|_{r=0} = \lim_{r \searrow 0} \frac{w(x, r, t) - w(x, 0, t)}{r} = \frac{\partial w}{\partial r}(x, 0, t)$$

meaning that the extended function $\tilde{w}$ is $C^1$ in $r = 0$ $\sim$ $\tilde{w} \in C^1(\mathbb{R}^3 \times \mathbb{R} \times [0, \infty); \mathbb{R})$. $\tilde{w}$ is also $C^2$, to that: note that $w(x, 0, t) = 0$ for all $t \geq 0$ and so $w_t(x, 0, t) = 0$ for all $t \geq 0$ and since $w$ satisfies the wave equation we conclude $w_{rr}(x, 0, t) = 0$. Furthermore, for $r < 0$:

$$\tilde{w}_{rr}(x, r, t) = \frac{\partial^2}{\partial r^2} \tilde{w}(x, r, t) = -\frac{\partial^2}{\partial r^2} w(x, -r, t) \quad \text{wave eq.} \quad -\frac{\partial^2}{\partial r^2} w(x, -r, t) = \frac{\partial^2}{\partial r^2} \hat{w}(x, r, t) = \hat{w}_{tt}(x, r, t)$$

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so that \( \tilde{w} \) is a \( C^2 \)-solution of the wave equation in 1D:

\[
\tilde{w}_{tt} - \tilde{w}_{rr} \equiv 0, \quad \forall x \in \mathbb{R}, \forall t \geq 0, \forall r \in \mathbb{R}.
\]

Recall, that \( w \) had the initial values

\[
w(x, r, 0) = G(x, r) \quad \text{and} \quad w_t(x, r, 0) = H(x, r) \quad \forall x \in \mathbb{R}^3, \forall r \geq 0,
\]

where we have set

\[
G(x, r) := r \cdot (Mg)(x, r) \quad \text{and} \quad H(x, r) := r \cdot (Mh)(x, r).
\]

Considering here also the odd reflection across 0:

\[
\tilde{G}(x, r) := \begin{cases} 
G(x, r), & r \geq 0, \\
-G(x, -r), & r < 0,
\end{cases} = \begin{cases} 
r \cdot (Mg)(x, r), & r \geq 0, \\
r \cdot (Mg)(x, -r), & r < 0,
\end{cases}
\]

and

\[
\tilde{H}(x, r) := \begin{cases} 
H(x, r), & r \geq 0, \\
-H(x, -r), & r < 0,
\end{cases} = \begin{cases} 
r \cdot (Mh)(x, r), & r \geq 0, \\
r \cdot (Mh)(x, -r), & r < 0,
\end{cases}
\]

we obtain that for \( r < 0 \):

\[
\tilde{w}(x, r, 0) = -w(x, -r, 0) = -G(x, -r) = r \cdot (Mg)(x, r) = \tilde{G}(x, r),
\]

and

\[
\tilde{w}_t(x, r, 0) = -w_t(x, -r, 0) = -H(x, -r) = r \cdot (Mh)(x, r) = \tilde{H}(x, r).
\]

Summarizing, we have shown that the extended function \( \tilde{w} \) is a \( C^2 \)-solution of the Cauchy problem of the wave equation in 1d (for any fixed \( x \in \mathbb{R}^3 \), see also Exercise 36):

\[
\begin{cases}
\tilde{w}_{tt} - \tilde{w}_{rr} \equiv 0, & \forall t \geq 0, \forall r \in \mathbb{R}, \\
\tilde{w}(x, r, 0) = \tilde{G}(x, r), & \tilde{w}_t(x, r, 0) = \tilde{H}(x, r), & \forall r \in \mathbb{R},
\end{cases}
\]

with the extended initial data \( \tilde{G} \) and \( \tilde{H} \). By d’Alembert’s formula the unique solution of this problem is given by

\[
\tilde{w}(x, r, t) = \frac{1}{2} \left[ \tilde{G}(x, r - t) + \tilde{G}(x, r + t) + \int_{r-t}^{r+t} \tilde{H}(x, s) \, ds \right].
\]

Then:

\[
u(t, x) \overset{(6.9)}{=} \lim_{r \searrow 0} (Mu(t, \cdot))(x, r) = \lim_{r \searrow 0} \frac{w(x, r, t)}{r} = \lim_{r \searrow 0} \frac{1}{2} \left[ \frac{\tilde{G}(x, r - t) + \tilde{G}(x, r + t)}{r} + \frac{1}{r} \int_{r-t}^{r+t} \tilde{H}(x, s) \, ds \right]
\]

\[
= \lim_{r \searrow 0} \frac{1}{2} \left[ \frac{G(x, t) - G(x, t - r)}{r} + \frac{G(x, t + r) - G(x, t)}{r} + \frac{1}{r} \int_{t-r}^{t+r} H(x, s) \, ds - \frac{1}{r} \int_{0}^{0} -H(x, -s) \, ds \right]
\]

\[
= \frac{1}{2} \left[ \frac{\partial}{\partial t} G(x, t) + \frac{\partial}{\partial t} G(x, t) \right] + \lim_{r \searrow 0} \frac{1}{2} \left[ \frac{1}{r} \int_{t-r}^{t+r} H(x, s) \, ds - \frac{1}{r} \int_{0}^{0} H(x, s) \, ds \right]
\]

\[
= \frac{\partial}{\partial t} G(x, t) + \lim_{r \searrow 0} \frac{1}{2} \left[ \frac{1}{r} \int_{t-r}^{t+r} H(x, s) \, ds + \frac{1}{r} \int_{t-r}^{t+r} H(x, s) \, ds \right] = \frac{\partial}{\partial t} G(x, t) + H(x, t)
\]

\[
= \frac{\partial}{\partial t} \left[ t \cdot (Mg)(x, t) \right] + t \cdot (Mh)(x, t).
\]

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Furthermore, recall that
\[
\frac{\partial}{\partial t} (Mg)(x,t) = \frac{1}{4\pi} \int_{\partial B_1(0)} g(x+ty) \, dA(y) \quad g \in C^3(\mathbb{R}^3; \mathbb{R}) \text{ compact} \quad \frac{1}{4\pi} \int_{\partial B_1(0)} D_x g(x+ty) \, y \, dA(y)
\]
\[
= \left\{ y = \frac{z-x}{t} \right\} = \frac{1}{4\pi t^2} \int_{\partial B_1(x)} \langle \nabla g(z), \frac{z-x}{t} \rangle \, dA(z).
\]

All in all, we conclude with the desired form:
\[
u(t, x) = \frac{\partial}{\partial t} \left[ t \cdot (Mg)(x,t) \right] + t \cdot (Mh)(x,t) = (Mg)(x,t) + t \cdot \frac{\partial}{\partial t} (Mg)(x,t) + t \cdot (Mh)(x,t)
\]
\[
= \int_{\partial B_1(x)} g(z) \, dA(z) + t \cdot \frac{1}{4\pi t^2} \int_{\partial B_1(x)} \langle \nabla g(z), \frac{z-x}{t} \rangle \, dA(z) + t \cdot \int_{\partial B_1(x)} h(z) \, dA(z)
\]
\[
= \frac{1}{4\pi t^2} \int_{\partial B_1(x)} g(z) + \langle \nabla g(z), z-x \rangle + t \cdot h(z) \, dA(z).
\]

ad 2: we are left to show: this \(u\) satisfies the Cauchy problem: First, since \(g \in C^3(\mathbb{R}^3; \mathbb{R})\) and \(h \in C^2(\mathbb{R}^2; \mathbb{R})\) the function defined by Kirchhoff’s formula is \(C^2\). Then:
\[
u(t, x) = (Mg)(x,t) + t \cdot \frac{\partial}{\partial t} (Mg)(x,t) + t \cdot (Mh)(x,t) \xrightarrow{t \to 0} g(x),
\]
i.e. \(u(0, x) = g(x)\). Moreover, since
\[
\frac{\partial}{\partial t} (Mg)(x,t) \xrightarrow{(6.10)} \frac{1}{4\pi} \int_{\partial B_1(0)} \langle \nabla_x g(x+ty), y \rangle \, dA(y)
\]
\[
\xrightarrow{t \to 0} \frac{1}{4\pi} \int_{\partial B_1(0)} \langle \nabla_x g(x), y \rangle \, dA(y) = \frac{1}{4\pi} \int_{\partial B_1(0)} \langle \nabla_x g(x), \nu(y) \rangle \, dA(y)
\]
\[
div \text{thm} \xrightarrow{\text{div thm}} \frac{1}{4\pi} \int_{B_1(0)} \text{div} \, (\nabla_x g(x)) \, dy = 0,
\]
we obtain
\[
\frac{\partial}{\partial t} u(t, x) = 2 \cdot \frac{\partial}{\partial t} (Mg)(x,t) + t \cdot \frac{\partial^2}{\partial t^2} (Mg)(x,t) + (Mh)(x,t) + t \cdot \frac{\partial}{\partial t} (Mh)(x,t) \xrightarrow{t \to 0} h(x),
\]
meaning that \(u(0, x) = h(x)\), and that this \(u\) satisfies the initial conditions. It remains to show, that this \(u\) satisfies the wave equation. Indeed, for any function \(v \in C^2(\mathbb{R}^3; \mathbb{R})\), the function \(w(t, x) := t \cdot (Mv)(x,t)\) satisfies the wave equation: we have
\[
\frac{\partial^2}{\partial t^2} w(t, x) = \frac{\partial}{\partial t} \left[ (Mv)(x,t) + t \cdot \frac{\partial}{\partial t} (Mv)(x,t) \right] = 2 \frac{\partial}{\partial t} (Mv)(x,t) + t \cdot \frac{\partial^2}{\partial t^2} (Mv)(x,t) \xrightarrow{n=3} t \cdot \Delta_x (Mv)(x,t)
\]
\[
= \Delta_x w(t, x).
\]
Furthermore, if \(\varphi \in C^3([0, \infty) \times \mathbb{R}^3; \mathbb{R})\) is a solution of the wave equation, then also \(\varphi_t\) satisfies the wave equation:
\[
\varphi_{tt} - \Delta_x \varphi = 0 \quad \equiv \quad 0 \equiv \left( \varphi_{tt} \right)_{t} - \left( \Delta_x \varphi \right)_{t} \equiv (\varphi_t)_{tt} - \Delta_x (\varphi_t).
\]

Summarizing, the function
\[
u(t, x) = \frac{\partial}{\partial t} \left[ t \cdot (Mg)(x,t) \right] + t \cdot (Mh)(x,t)
\]
solves wave eq. solves wave eq. solves wave eq.
is a solution of the wave equation. □
6.5 \( n = 2 \)

Problem: There does not exist a transformation (in 2d) to convert the Euler-Poisson-Darboux equation into a wave equation in 1d.

Solution: **Hadamard’s method of descent:**

Regard the initial value problem for \( n = 2 \) as problem for \( n = 3 \), in which the third spatial variable does not appear: Indeed, let \( u \in C^2([0, \infty) \times \mathbb{R}^2; \mathbb{R}) \) be a solution of

\[
\begin{cases}
u_{tt} - \Delta_x u \equiv 0, & \text{in } [0, \infty) \times \mathbb{R}^2, \\
u(0, x) = g(x), u_t(0, x) = h(x), & \forall x \in \mathbb{R}^2.
\end{cases}
\]

Define:

\[
\overline{u}(t, x_1, x_2, x_3) := u(t, x_1, x_2).
\]

Then:

\[
\frac{\partial^2}{\partial t^2} \overline{u}(t, x_1, x_2, x_3) = \frac{\partial^2}{\partial t^2} u(t, x_1, x_2)
\overset{\text{sol. wave eq. 2d}}{=} \Delta_{(x_1, x_2)} u(t, x_1, x_2) = \frac{\partial^2}{\partial x_1^2} u(t, x_1, x_2) + \frac{\partial^2}{\partial x_2^2} u(t, x_1, x_2)
\]

\[
= \frac{\partial^2}{\partial x_1^2} \overline{u}(t, x_1, x_2, x_3) + \frac{\partial^2}{\partial x_2^2} \overline{u}(t, x_1, x_2, x_3) + \frac{\partial^2}{\partial x_3^2} \overline{u}(t, x_1, x_2, x_3) = \Delta_{(x_1, x_2, x_3)} \overline{u}(t, x_1, x_2, x_3),
\]

i.e. \( \overline{u} \) is a solution of the wave equation in 3d. Moreover, \( \overline{u} \) satisfies the initial conditions:

\[
\overline{u}(0, x_1, x_2, x_3) = u(0, x_1, x_2) = g(x_1, x_2) =: \tilde{g} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\]

and

\[
\overline{u}_t(0, x_1, x_2, x_3) = u_t(0, x_1, x_2) = h(x_1, x_2) =: \tilde{h} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
\]

According to Kirchhoff’s formula we know that the unique solution is given by:

\[
\overline{u}(t, x_1, x_2, x_3) = \frac{1}{4\pi t^2} \int_{\partial B_t(x)} \tilde{g}(z) + \left( \nabla \tilde{g}(z), \begin{pmatrix} z_1 - x_1 \\ z_2 - x_2 \\ z_3 - x_3 \end{pmatrix} \right) + t \cdot \tilde{h}(z) \, dA(z),
\]

and in particular for \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2: \)

\[
u(t, x) = \overline{u}(t, x_1, x_2, 0)
\]

\[
= \frac{1}{4\pi t^2} \int_{\partial B_t(x)} \tilde{g}(z) + \left( \nabla \tilde{g}(z), \begin{pmatrix} z_1 - x_1 \\ z_2 - x_2 \\ z_3 \end{pmatrix} \right) + t \cdot \tilde{h}(z) \, dA(z).
\]

We now consider the following parametrization of the upper hemisphere: for \( y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in B_t(x) \subset \mathbb{R}^2 \) by the graph of

\[
\sqrt{t^2 - |x - y|^2}, \text{ i.e. by } \psi(y) = \begin{pmatrix} y_1 \\ y_2 \\ \sqrt{t^2 - |x - y|^2} \end{pmatrix}.
\]

Figure 21: \( x \in \mathbb{R}^2 \) is seen as \( \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \) in \( \mathbb{R}^3 \).
In order to compute the change of the surface measure, we consider the tangent vectors at $\psi(y)$:

$$\tau_1(y) = \frac{\partial \psi}{\partial y_1}(y) = \begin{pmatrix} 1 \\ 0 \\ \frac{x_1-y_1}{\sqrt{t^2-|x-y|^2}} \end{pmatrix} \quad \text{and} \quad \tau_2(y) = \frac{\partial \psi}{\partial y_2}(y) = \begin{pmatrix} 0 \\ 1 \\ \frac{x_2-y_2}{\sqrt{t^2-|x-y|^2}} \end{pmatrix},$$

they span a parallelogram of area

$$|\tau_1(y) \times \tau_2(y)| = \left| \begin{pmatrix} 1 \\ 0 \\ \frac{x_1-y_1}{\sqrt{t^2-|x-y|^2}} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \frac{x_2-y_2}{\sqrt{t^2-|x-y|^2}} \end{pmatrix} \right| = \frac{\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}}{t^2 - |x-y|^2} + 1$$

Then we obtain for the surface integral

$$\int_{\partial B_t(x)} \gamma(z) dA(z) = \int_{B_t(x)} \gamma(\psi(y)) |\tau_1(y) \times \tau_2(y)| \, dy = \int_{B_t(x)} \frac{\gamma(\psi(y)) \cdot t}{\sqrt{t^2 - |x-y|^2}} \, dy.$$

Hence:

$$u(t, x) = \frac{1}{4\pi t^2} \int_{\partial B_t(x)} \bar{g}(z) + \left( \nabla \bar{g}(z), \begin{pmatrix} z_1-x_1 \\ z_2-x_2 \\ z_3 \end{pmatrix} \right) \cdot t \cdot \bar{h}(z) \, dA(z)$$

$$+ \frac{1}{4\pi t^2} \int_{\partial B_t(x)} \bar{g}(z) + \left( \nabla \bar{g}(z), \begin{pmatrix} z_1-x_1 \\ z_2-x_2 \\ z_3 \end{pmatrix} \right) \cdot t \cdot \bar{h}(z) \, dA(z)$$

$$= \frac{1}{4\pi t^2} \int_{B_t(x)} \left\{ \bar{g} \left( \frac{y_1}{\sqrt{t^2 - |x-y|^2}} \right) + \left( \partial_1 \bar{g} \left( \frac{y_1}{\sqrt{t^2 - |x-y|^2}} \right), \frac{y_1-x_1}{\sqrt{t^2 - |x-y|^2}} \right) \right\} \frac{t}{\sqrt{t^2 - |x-y|^2}} \, dy$$

$$+ \left\{ \bar{g} \left( \frac{y_2}{\sqrt{t^2 - |x-y|^2}} \right) + \left( \partial_2 \bar{g} \left( \frac{y_2}{\sqrt{t^2 - |x-y|^2}} \right), \frac{y_2-x_2}{\sqrt{t^2 - |x-y|^2}} \right) \right\} \frac{t}{\sqrt{t^2 - |x-y|^2}} \, dy$$

$$= \frac{1}{2\pi t} \int_{B_t(x)} \{ g(y) + \langle \nabla y \bar{g}(y), y-x \rangle_{L^2} + t \cdot h(y) \} \frac{1}{\sqrt{t^2 - |x-y|^2}} \, dy.$$

This is Poisson’s formula and represents the unique solution of the Cauchy problem of the wave equation in 2d. Indeed, we have derived (6.12) assuming that $u$ is a $C^2$-solution, also the converse holds true
Theorem 6.7 (Solution to the wave equation in 2d). Let \( g \in C^3(\mathbb{R}^2; \mathbb{R}) \), \( h \in C^2(\mathbb{R}^2; \mathbb{R}) \). Then \( u \) defined by (6.12) satisfies

i) \( u \in C^2((0, \infty) \times \mathbb{R}^2; \mathbb{R}) \),

ii) \( u_{tt} - \Delta u \equiv 0 \) in \((0, \infty) \times \mathbb{R}^2, \)

iii) \( \lim_{t \to 0, x \in \mathbb{R}^2} u(t, x) = g(b) \) and \( \lim_{t \to 0, x \in \mathbb{R}^2} u(t, x) = h(b) \) for all \( b \in \mathbb{R}^2 \).

Proof. Apply Theorem 6.5 and Poisson’s formula. \( \square \)

Remark 6.8. In contrast to Kirchhoff’s formula (solution of the wave equation in 3d), in Poisson’s formula for the solution of the wave equation in 2d we integrate over the whole disk \( B_1(x) \) and not just on \( \partial B_1(x) \). \( \sim \) Huygen’s principle in 2d

Remark 6.9. In order to obtain formula’s for the solutions of the Cauchy problems of the wave equation in odd dimensions proceed inductively, see Exercise 43 for \( n = 5 \), and in even dimensions apply the method of descents, see also EVANS: PDE, Sec. 2.4.1 d, e.

## 6.6 Inhomogeneous wave equation

Consider the inhomogeneous wave equation

\[
\begin{align*}
\begin{cases}
 u_{tt}(t, x) - \Delta u(t, x) &= f(t, x), & \text{in } [0, \infty) \times \mathbb{R}^n, \\
 u(0, x) = 0, \ u_t(0, x) = 0, & \forall \ x \in \mathbb{R}^n.
\end{cases}
\end{align*}
\]  
\tag{6.13}

\( \sim \) Duhamel’s principle (cf. Section 4.4): for fixed \( s \in [0, t] \); consider the shifted homogeneous problem:

\[
\begin{align*}
\begin{cases}
 \frac{\partial^2}{\partial t^2} u^s(t, x) - \Delta u^s(t, x) &= 0, & \text{in } [s, \infty) \times \mathbb{R}^n, \\
 u^s(s, x) = 0, \ \frac{\partial}{\partial t} u^s(s, x) &= f(s, x), & \forall \ x \in \mathbb{R}^n.
\end{cases}
\end{align*}
\]  
\tag{6.14}

We have seen the explicit expression of the unique solution of (6.14) for \( n \in \{1, 2, 3\} \). More precisely, we know the solution of

\[
\begin{align*}
\begin{cases}
 \varphi_{tt} - \Delta_x \varphi &= 0, & \text{in } [0, \infty) \times \mathbb{R}^n, \\
 \varphi(0, x) = \gamma(x), \ \varphi_t(0, x) = \beta(x), & \forall \ x \in \mathbb{R}^n,
\end{cases}
\end{align*}
\]

for \( n \in \{1, 2, 3\} \) and sufficiently smooth functions \( \beta(\cdot), \gamma(\cdot) \). Then: \( \psi(t, x) := \varphi(t - s, x) \) solves the shifted initial value problem

\[
\begin{align*}
\begin{cases}
 \psi_{tt} - \Delta_x \psi &= 0, & \text{in } [s, \infty) \times \mathbb{R}^n, \\
 \psi(s, x) = \gamma(x), \ \psi_t(s, x) = \beta(x), & \forall \ x \in \mathbb{R}^n.
\end{cases}
\end{align*}
\]

Thus, let us denote the solution of (6.14) by \( u^s(\cdot, \cdot) \). We claim that the solution of (6.13) is given by

\[
u(t, x) := \int_0^t u^s(t, x) \, ds.
\]

Let indeed \( u^s \in C^2([s, \infty) \times \mathbb{R}^n; \mathbb{R}) \) be a solution of (6.14), then

\[
\frac{\partial}{\partial t} u^s(t, x) = \int_0^t \frac{\partial}{\partial t} u^s(t, x) \, ds = u^t(t, x) + \int_0^t \frac{\partial}{\partial t} u^s(t, x) \, ds = \frac{\partial}{\partial t} u^s(t, x) \, ds.
\]

Hence:

\[
\frac{\partial^2}{\partial t^2} u^s(t, x) = \int_0^t \frac{\partial^2}{\partial t^2} u^s(t, x) \, ds = \frac{\partial}{\partial t} u^s(t, x) + \int_0^t \frac{\partial^2}{\partial t^2} u^s(t, x) \, ds \overset{(6.14)}{=} f(t, x) + \int_0^t \Delta_x u^s(t, x) \, ds
\]
\[ = f(t, x) + \Delta_x \int_0^t u^s(t, x) \, ds = f(t, x) + \Delta_x u(t, x). \]

Moreover:

\[ u(0, x) = \int_0^0 u^s(0, x) \, ds = 0 \quad \text{and} \quad u_t(0, x) = \int_0^0 \frac{\partial}{\partial t} u^s(0, x) \, ds = 0, \]

i.e. \( u \) satisfies (6.13).

**Question:** How to solve a general inhomogeneous Cauchy problem:

\[
\begin{cases}
  u_{tt}(t, x) - \Delta_x u(t, x) = f(t, x), & \text{in } [0, \infty) \times \mathbb{R}^n,
  \\
  u(0, x) = g(x), \quad u_t(0, x) = h(x), & \forall \, x \in \mathbb{R}^n,
\end{cases}
\]

for \( n \in \{1, 2, 3\} \). This problem is a linear one \( \leadsto \) decompose and search for solutions of:

\[
\begin{align*}
  v_{tt}(t, x) - \Delta_x v(t, x) & \equiv 0 & \text{in } [0, \infty) \times \mathbb{R}^n, \\
  v(0, x) = g(x), \quad v_t(0, x) = h(x), & \forall \, x \in \mathbb{R}^n, \\
  w_{tt}(t, x) - \Delta_x w(t, x) & = f(t, x) & \text{in } [0, \infty) \times \mathbb{R}^n, \\
  w(0, x) = 0, \quad w_t(0, x) = 0, & \forall \, x \in \mathbb{R}^n,
\end{align*}
\]

homogeneous problem: \( \exists \) formula \quad \text{Duhamel’s principle}

Then: \( u := v + w \) solves the general inhomogeneous problem.

**Examples:**

\( n = 1 \):

\[ v(t, x) = \frac{1}{2} \left[ g(x - t) + g(x + t) + \int_{x-t}^{x+t} h(y) \, dy \right] \quad \text{d’Alembert} \]

\[ w^*(t, x) = \frac{1}{2} \int_{x-(t-s)}^{x+(t-s)} f(s, y) \, dy \quad \Rightarrow \quad w(t, x) = \frac{1}{2} \int_0^t f(s, y) \, dy \, ds = \int \left\{ s = t - \tau \right\} \frac{1}{2} \int_{x-\tau}^{x+\tau} f(t - \tau, y) \, dy \, d\tau 
\]

\[ = \frac{1}{2} \int_0^t f(t - \tau, y) \, dy \, d\tau \]

\( n = 2 \):

\[ v(t, x) = \frac{1}{2\pi t} \int_{B_1(x)} g(y) + \left( \nabla g(y), y - x \right)_{\mathbb{R}^2} + t \cdot h(y) \sqrt{t^2 - |x - y|^2} \, dy \quad \text{Poisson} \]

\[ w^*(t, x) = \frac{1}{2\pi(t-s)^\frac{3}{2}} \int_{B_{t-s}(x)} \frac{(t-s) \cdot f(s, y) \sqrt{(t-s)^2 - |x-y|^2}}{dy} \quad \Rightarrow \quad w(t, x) = \frac{1}{2\pi} \int_0^t \int_{B_{t-s}(x)} \frac{f(s, y)}{\sqrt{(t-s)^2 - |x-y|^2}} \, dy \, ds 
\]

\[ = \int \left\{ s = t - \tau \right\} \frac{1}{2\pi} \int_0^t \int_{B_{t-s}(x)} \frac{f(t - \tau, y)}{\sqrt{\tau^2 - |x-y|^2}} \, dy \, d\tau \]

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\[ n = 3: \quad v(t, x) = \frac{1}{4\pi t^2} \int_{\partial B_t(x)} g(y) + \langle \nabla g(y), y - x \rangle_{\mathbb{R}^3} + t \cdot h(y) \, dA(y) \quad \text{Kirchhoff} \]

\[ w^a(t, x) = \frac{1}{4\pi (t-s)^2} \int_{\partial B_{t-s}(x)} (t-s) \cdot f(s, y) \, dA(y) \implies w(t, x) = \frac{1}{4\pi} \int_0^t \frac{1}{t-s} \int_{\partial B_{t-s}(x)} f(s, y) \, dA(y) \, ds \]

\[ = \left\{ s = t-\tau \right\} = \frac{1}{4\pi} \int_0^t \int_{\partial B_s(x)} \frac{f(t-\tau, y)}{\tau} \, dA(y) \, d\tau \]

\[ = \frac{1}{4\pi} \int_{B_t(x)} \frac{f(t-x-y)}{|x-y|} \, dy. \]

Cavalieri’s principle or Kirchhoff’s domain is the domain by characteristics.

\[ 0 \equiv \left\langle \begin{pmatrix} 1 \\ b \end{pmatrix}, \begin{pmatrix} \frac{\partial u}{\partial t} \\ \nabla_x u \end{pmatrix} \right\rangle_{\mathbb{R}^{n+1}} = \left\langle \begin{pmatrix} 1 \\ b \end{pmatrix}, \nabla (t, x) u \right\rangle_{\mathbb{R}^{n+1}}, \]

i.e. the directional derivative of \( u \) vanishes in the direction \( \begin{pmatrix} 1 \\ b \end{pmatrix} \), meaning that \( u \) is constant along curves (here: lines) in the direction \( \begin{pmatrix} 1 \\ b \end{pmatrix} \). These curves are parametrized by \( \begin{pmatrix} 1 \\ b \end{pmatrix} s + \begin{pmatrix} 0 \\ a \end{pmatrix} \) for fixed \( a \in \mathbb{R}^n \) and parameter \( s \in [0, \infty) \).

In order to compute the value of the solution \( u \) at a point \( \begin{pmatrix} t \\ x \end{pmatrix} \) proceed in the following way: find the corresponding characteristic and go back to the initial data:

\[ \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ b \end{pmatrix} \bar{s} + \begin{pmatrix} 0 \\ a \end{pmatrix} \implies \bar{s} = t \]

\[ \implies \bar{\pi} = x - b\bar{s} = x - bt \]

\[ \implies u(t, x) = g(\bar{\pi}) = g(x - bt). \]

Note: The initial data at time \( t = 0 \) can also be seen as boundary data since the problem is considered on \([0, \infty) \times \mathbb{R}^n \subset \mathbb{R}^{n+1}\). This boundary coincides with each characteristic only in one point and all characteristics form a foliation of the domain (here: space-time). Important, since this guarantees that each interior point is reached precisely once, so that the solution can be reconstructed uniquely.

Question: How can we find characteristics?

Consider an \( n \)-th order pde: \( F(\nabla u, u, x) = 0 \) in \( \Omega \), where \( \Omega \text{ (open)} \subseteq \mathbb{R}^n \), \( F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}, (p, z, x) \rightarrow F(p, z, x) \), is sufficiently smooth. In addition, assume that on \( \Gamma \subseteq \partial \Omega \) the boundary values are prescribed: \( u(b) = g(b) \) for all \( b \in \Gamma \), with sufficiently smooth function \( g(\cdot) \):

\[ \begin{cases} F(\nabla u, u, x) = 0 \quad \text{in } \Omega, \\ u = g \quad \text{on } \Gamma. \end{cases} \quad (7.1) \]

Figure 24: Partially prescribed data.

Figure 23: A foliation of the domain by characteristics.
Consider a curve described parametrically by \( \pi : I \rightarrow \Omega \) with \( \pi(s) = \begin{pmatrix} x_1(s) \\ \vdots \\ x_n(s) \end{pmatrix} \) and adequate interval \( I \subseteq \mathbb{R} \).

Furthermore, we set \( \pi(s) := u(\pi(s)) \) (values of \( u \) along the curve \( \pi(\cdot) \)) and \( \overline{p}(s) := \nabla u(\pi(s)) \) (values of the gradient of \( u \) along the curve \( \pi(\cdot) \)), whereby \( u \) denotes a \( C^2 \)-solution of (7.1). Then

\[
\frac{d}{ds} \overline{p}(s) \bigg|_{\text{chain rule}} = D\nabla u(\pi(s)) \frac{d}{ds} \pi(s).
\]

Moreover, a differentiation of \( F(\nabla u, u, x) \equiv 0 \) with respect to \( x \) gives by chain rule:

\[
0 \equiv D_x [F(\nabla u, u, x)] = (D_p F) (\nabla u, u, x) D\nabla u + (D_z F) (\nabla u, u, x) Du + (D_x F) (\nabla u, u, x).
\]

(7.2)

Assume now in addition, that

\[
\frac{d}{ds} \pi(s) = \nabla_p F(\overline{p}(s), \pi(s), \nabla \pi(s)),
\]

then, the transposed (7.2) along \( \pi(s) \) reads:

\[
0 \equiv D\nabla u(\pi(s)) \nabla_p F(\overline{p}(s), \pi(s), \nabla \pi(s)) + \nabla u(\pi(s)) D_z F(\overline{p}(s), \pi(s), \nabla \pi(s)) + \nabla_x F(\overline{p}(s), \pi(s), \nabla \pi(s))
\]

\[
= D\nabla u(\pi(s)) \frac{d}{ds} \pi(s) + \overline{p}(s) D_z F(\overline{p}(s), \pi(s), \nabla \pi(s)) + \nabla_x F(\overline{p}(s), \pi(s), \nabla \pi(s))
\]

\[
= \frac{d}{ds} \overline{p}(s) + \overline{p}(s) \frac{\partial}{\partial z} F(\overline{p}(s), \pi(s), \nabla \pi(s)) + \nabla_x F(\overline{p}(s), \pi(s), \nabla \pi(s))
\]

\[
\Rightarrow \frac{d}{ds} \overline{p}(s) = - \frac{\partial}{\partial z} F(\overline{p}(s), \pi(s), \nabla \pi(s)) \overline{p}(s) + \nabla_x F(\overline{p}(s), \pi(s), \nabla \pi(s)).
\]

Finally, differentiating \( \pi(s) = u(\pi(s)) \) with respect to \( s \) we obtain:

\[
\frac{d}{ds} \pi(s) = D_x u(\pi(s)) \frac{d}{ds} \pi(s) = \left[ \overline{p}(s) \right]^\top \nabla_p F(\overline{p}(s), \pi(s), \nabla \pi(s)) = \langle \nabla_p F(\overline{p}(s), \pi(s), \nabla \pi(s)), \overline{p}(s) \rangle_{\mathbb{R}^n}
\]

Hence, we obtain a system of \((2n + 1)\) first order o.d.e.'s:

\[
\begin{cases}
\dot{\pi}(s) = \nabla_p F(\overline{p}(s), \pi(s), \nabla \pi(s)), \\
\dot{\overline{p}}(s) = - \frac{\partial}{\partial z} F(\overline{p}(s), \pi(s), \nabla \pi(s)) \overline{p}(s) + \nabla_x F(\overline{p}(s), \pi(s), \nabla \pi(s)), \\
\dot{\pi}(s) = \langle \nabla_p F(\overline{p}(s), \pi(s), \nabla \pi(s)), \overline{p}(s) \rangle_{\mathbb{R}^n}.
\end{cases}
\]

The solutions \( \begin{pmatrix} \overline{p}(\cdot) \\ \pi(\cdot) \end{pmatrix} : I \rightarrow \mathbb{R}^n \times \mathbb{R} \times \Omega \) are called characteristics, and will refer to \( \pi(\cdot) \) as the projected characteristic. We have shown:

**Theorem 7.1 (Structure of characteristic o.d.e).** Let \( u \in C^2(\Omega; \mathbb{R}) \) solve the first order p.d.e \( F(\nabla u, u, x) \equiv 0 \) in \( \Omega \). If \( \pi(\cdot) \) solves the o.d.e

\[
\dot{\pi}(s) = \nabla_p F(\overline{p}(s), \pi(s), \nabla \pi(s)),
\]

where we have set \( \overline{p}(s) := \nabla u(\pi(s)) \) and \( \pi(s) := u(\pi(s)) \) then \( \overline{p}(\cdot) \) and \( \pi(\cdot) \) solve:

\[
\dot{\overline{p}}(s) = - \frac{\partial}{\partial z} F(\overline{p}(s), \pi(s), \nabla \pi(s)) \overline{p}(s) + \nabla_x F(\overline{p}(s), \pi(s), \nabla \pi(s)),
\]

\[
\dot{\pi}(s) = \langle \nabla_p F(\overline{p}(s), \pi(s), \nabla \pi(s)), \overline{p}(s) \rangle_{\mathbb{R}^n}
\]

for all \( s \) such that \( \pi(s) \in \Omega \).
Examples in 2d:

- reconstruct solution of semilinear problem:

\[
\begin{aligned}
&x_1 u_{x_2} - x_2 u_{x_1} = u \quad \text{in } (0, \infty) \times (0, \infty), \\
u(\gamma, 0) = g(\gamma) \quad \text{for all } \gamma \in (0, \infty).
\end{aligned}
\]

We have \( F(p, z, x) = x_1 p_2 - x_2 p_1 - z \), so that \( \nabla_p F = \left( -\frac{x_2}{x_1}, \frac{\partial}{\partial z} \right) \), \( \frac{\partial}{\partial z} F = -1 \), \( \nabla_z F = \left( p_2, -p_1 \right) \). Hence, we obtain the system (whereby we omit the overlines in the following calculations):

\[
\begin{aligned}
\dot{x}_1(s) &= -x_2(s), \\
\dot{x}_2(s) &= x_1(s), \\
\dot{p}_1(s) &= p_1(s) - p_2(s), \\
\dot{p}_2(s) &= p_2(s) + p_1(s), \\
\dot{s}(s) &= -x_2(s)p_1(s) + x_1(s)p_2(s) F(p, z, x) = 0 z(s).
\end{aligned}
\]

Thus, \( \ddot{x}_1(s) = -\ddot{x}_2(s) = -x_1(s) \) and \( \ddot{x}_2(s) = \ddot{x}_1(s) = -x_2(s) \), so that \( x_1(s) = \alpha \cos(s) \) and \( x_2(s) = \alpha \sin(s) \) for some \( \alpha > 0 \) and \( s \in (0, \pi) \). Furthermore, by the last equation we deduce \( z(s) = \beta e^\alpha \). By the prescribed data we obtain:

\[
\beta = \beta e^0 = z(0) = u(x_1(0), x_2(0)) = u(\alpha, 0) = g(\alpha).
\]

All in all, we have

\[
z(s) = g(\alpha) e^\alpha, \quad x_1(s) = \alpha \cos(s), \quad x_2(s) = \alpha \sin(s).
\]

In order to reconstruct the solution, find for an arbitrary point \((\xi_1, \xi_2) \in (0, \infty) \times (0, \infty)\) values \( \overline{\alpha} > 0 \) and \( \overline{\alpha} \in (0, \pi) \) such that

\[
(\xi_1, \xi_2) = (x_1(\overline{\alpha}), x_2(\overline{\alpha})) = (\overline{\alpha} \cos(\overline{\alpha}), \overline{\alpha} \sin(\overline{\alpha}))
\]

\[
\Rightarrow \overline{\alpha} = \sqrt{\xi_1^2 + \xi_2^2} \quad \text{and} \quad \tan(\overline{\alpha}) = \frac{\xi_2}{\xi_1}, \quad \overline{\alpha} \in (0, \frac{\pi}{2})
\]

so that we recover the solution

\[
u(\xi_1, \xi_2) = u(x_1(\overline{\alpha}), x_2(\overline{\alpha})) = z(\overline{\alpha}) = g(\overline{\alpha}) e^\overline{\alpha} = g \left( \sqrt{\xi_1^2 + \xi_2^2} \right) e^{\arctan \left( \frac{\xi_2}{\xi_1} \right)}
\]

- reconstruct solution of semilinear Cauchy problem, whereby data is prescribed on an arbitrary noncharacteristic curve (i.e. this curve is nowhere tangent to the projected characteristics):

\[
\begin{aligned}
u_{x_1} + x_1 u_{x_2} = u \quad \text{in } \mathbb{R}^2, \\
u(1, \gamma) = h(\gamma) \quad \text{for all } \gamma \in \mathbb{R}.
\end{aligned}
\]

We have \( F(p, z, x) = p_1 + x_1 p_2 - z \), so that \( \nabla_p F = \left( \frac{1}{x_1}, \frac{\partial}{\partial z} \right) \), \( \frac{\partial}{\partial z} F = -1 \), \( \nabla_z F = \left( p_2, 0 \right) \). Hence, we obtain the system

\[
\begin{aligned}
\dot{x}_1(s) &= 1, \\
\dot{x}_2(s) &= x_1(s), \\
\dot{z}(s) &= \left( \begin{array}{c} 1 \\ x_1(s) \end{array} \right) - \begin{pmatrix} p_1(s) \\ p_2(s) \end{pmatrix} F(p, z, x) = 0 z(s).
\end{aligned}
\]

\[
\Rightarrow \begin{cases}
x_1(s) = s + \beta, \\
x_2(s) = \frac{1}{2}s^2 + \beta s + \delta, \\
z(s) = \alpha e^\alpha.
\end{cases}
\]
The data is prescribed for \( s = 0 \):

\[
\begin{aligned}
1 &= x_1(0) = \beta, \\
\gamma &= x_2(0) = \delta, \\
h(\gamma) &= z(0) = \alpha,
\end{aligned}
\]

\[
\begin{aligned}
x_1(s) &= s + 1, \\
x_2(s) &= \frac{1}{2}s^2 + s + \gamma, \\
z(s) &= h(\gamma)e^s.
\end{aligned}
\]

For an arbitrary point \((\xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}\) find \(\pi \in \mathbb{R}\) and \(\tau \in \mathbb{R}\) such that:

\[
\begin{aligned}
(\xi_1, \xi_2) &= (x_1(\pi), x_2(\pi)) = (\pi + 1, \frac{1}{2}\pi^2 + \pi + \tau) \\
\Rightarrow \quad \pi &= \xi_1 - 1 \\
\Rightarrow \quad \tau &= \xi_2 - \frac{1}{2}\pi^2 - \pi = \xi_2 - \frac{1}{2}(\xi_1 - 1)^2 - (\xi_1 - 1)
\end{aligned}
\]

Hence:

\[
\begin{aligned}
u(\xi_1, \xi_2) &= u(x_1(\pi), x_2(\pi)) = z(\pi) = h(\tau)e^\pi \\
&= h\left(\xi_2 - \frac{1}{2}(\xi_1 - 1)^2 - (\xi_1 - 1)\right)e^{\xi_1 - 1}.
\end{aligned}
\]

- reconstruct solution of a fully nonlinear problem:

\[
\begin{aligned}
u_{x_1^2} + u_{x_2^2} &= 1 \quad \text{in } \mathbb{R}^2 \setminus \{0\}, \\
\nu|_{\partial B_1(0)} &= 0.
\end{aligned}
\]

We have \(F(p, z, x) = |p|^2 - 1\), so that \(\nabla_p F = p, \quad \frac{\partial}{\partial x} F = 0, \quad \nabla_z F = 0\) and we obtain the system

\[
\begin{aligned}
\dot{x}_1(s) &= 2p_1(s), \\
\dot{x}_2(s) &= 2p_2(s), \\
\dot{p}_1(s) &= 0, \\
\dot{p}_2(s) &= 0, \\
\phi(s) &= 2(\overline{p}(s), p(s)) F(\overline{p}, \pi, \pi) = 2.
\end{aligned}
\]

\[
\begin{aligned}
x_1(s) &= 2s \cos(\psi) + \gamma, \\
x_2(s) &= 2s \sin(\psi) + \delta, \\
z(s) &= 2s + \epsilon.
\end{aligned}
\]

The data is prescribed on \(\partial B_1(0)\), i.e., for \(\phi \in [0, 2\pi]\) we have:

\[
\begin{aligned}
\cos(\phi) &= x_1(0) = \gamma, \\
\sin(\phi) &= x_2(0) = \delta, \\
0 &= z(0) = \epsilon.
\end{aligned}
\]

Moreover, \(u\) is constant along \(\partial B_1(0)\), meaning that

\[
\left\langle \nabla u \right|_{\partial B_1(0)} \tau \right \rangle = 0,
\]

whereby the tangent vector \(\tau\) at a boundary point \(\left(\frac{\cos(\phi)}{\sin(\phi)}\right)\) is given by:

\[
\tau \left(\frac{\cos(\phi)}{\sin(\phi)}\right) = \left(-\sin(\phi), \cos(\phi)\right), \quad \text{so that}
\]

Figure 26: Foliation by characteristics.

Figure 27: Foliation by characteristics.
\[ 0 = \langle p(0), \tau \rangle = \left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \end{pmatrix}, \begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \end{pmatrix} \right\rangle = -\cos(\psi)\sin(\phi) + \sin(\psi)\cos(\phi) \]

First case: \( \psi - \phi = 0 \), then we have:

\[
\begin{align*}
\begin{cases}
x_1(s) = (2s + 1) \cos(\phi), \\
x_2(s) = (2s + 1) \sin(\phi), \\
z(s) = 2s.
\end{cases}
\end{align*}
\]

Hence: for an arbitrary point \((\xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R} \setminus \{(0, 0)\}\) find \(\phi \in [0, 2\pi), \varpi \in (-\frac{1}{2}, +\infty)\) such that

\[
\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = (2\varpi + 1) \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix}
\Rightarrow \ldots
\]

and \(\xi_1^2 + \xi_2^2 = (2\varpi + 1)^2 = (z(\varpi) + 1)^2 = (u(\xi_1, \xi_2) + 1)^2 \Rightarrow u(\xi_1, \xi_2) = -1 \pm \sqrt{\frac{\xi_1^2}{\xi_2}}\).

Since the prescribed data vanishes on the unit circle, we obtain as solution: \(u(\xi_1, \xi_2) = -1 + \sqrt{\xi_1^2 + \xi_2^2}\).

Second case: \( \psi - \phi = \pi \), then \(\cos(\psi) = \cos(\pi + \phi) = -\cos(\phi)\) and \(\sin(\psi) = \sin(\pi + \phi) = -\sin(\phi)\), so that we obtain:

\[
\begin{align*}
\begin{cases}
x_1(s) = (-2s + 1) \cos(\phi), \\
x_2(s) = (-2s + 1) \sin(\phi), \\
z(s) = 2s.
\end{cases}
\end{align*}
\]

As above we conclude \(\xi_1^2 + \xi_2^2 = (-2\varpi + 1)^2 = (-u(\xi_1, \xi_2) + 1)^2 \Rightarrow u(\xi_1, \xi_2) = 1 \pm \sqrt{\xi_1^2 + \xi_2^2}\).

Since the prescribed data vanishes on the unit circle, we obtain as solution: \(u(\xi_1, \xi_2) = 1 - \sqrt{\xi_1^2 + \xi_2^2}\).

Thus, our solution is not uniquely determined (only up to a sign):

\[
u(x_1, x_2) = \pm(1 - \sqrt{x_1^2 + x_2^2}) \quad \text{or} \quad u(x) = \pm(1 - |x|).
\]

Compatibility conditions: data prescribed on flat space.

Let \(\Omega'(\text{open}) \subseteq \mathbb{R}^{n-1}\), set \(\Omega := \Omega' \times \mathbb{R}, \Gamma := \Omega' \times \{0\} \subset \mathbb{R}^n\) and consider the problem

\[
\begin{align*}
\begin{cases}
F(\nabla u, u, x) & \equiv 0 \quad \text{in } \Omega, \\
u(x', 0) = g(x') & \forall \ x' \in \Omega'.
\end{cases}
\end{align*}
\]

Since our characteristics pass through \(\Gamma\) for \(s = 0\) we have:

\[
\Gamma \ni \varpi(0) = \begin{pmatrix} x_1(0) \\ \vdots \\ x_{n-1}(0) \\ 0 \end{pmatrix} = \begin{pmatrix} \varpi'(0) \\ 0 \end{pmatrix}, \quad \varpi'(0) \in \Omega',
\]

\[
\varpi(0) = u(\varpi(0)) = u \begin{pmatrix} \varpi'(0) \\ 0 \end{pmatrix} = g(\varpi'(0)),
\]

\[
\Rightarrow \quad p_i(0) = \frac{\partial}{\partial x_i} u(\varpi(0)) = \frac{\partial}{\partial x_i} g(\varpi'(0)) \quad \text{for } i \in \{1, \ldots, n-1\}.
\]

![Figure 28: \(\Omega' \subset \mathbb{R}^{n-1}\) and \(\Gamma = \Omega' \times \{0\} \subset \mathbb{R}^n\).](image)
Thus: The prescribed conditions for our ODE:

\[ p(0) = q, \quad \zeta(0) = \zeta, \quad \xi(0) = \xi' \]

with \((q, \zeta, \xi') \in \mathbb{R}^n \times \mathbb{R} \times \Omega'\) must satisfy:

\[
\begin{cases}
\zeta = g(\xi'), \\
q_i = \frac{\partial}{\partial x_i} g(\xi') \quad i \in \{1, \ldots, n-1\}, \\
F(q, \zeta, \xi') = 0.
\end{cases}
\]

\[ \Rightarrow \text{compatibility conditions} \]

A triple \((q, \zeta, \xi')\) satisfying these compatibility conditions is called \textit{admissible}.

Now: reconstruct solution \(u\) of initial first order PDE new \(\xi'\) by solving the characteristic ODE’s. Thus, given a point \(y' \in \Omega'\) (close to \(\xi'\)) solve

\[
\begin{cases}
\dot{x}(s) = \nabla_p F(p(s), \zeta(s), \xi(s)) \\
\dot{p}(s) = -\frac{\partial}{\partial z} F(p(s), \zeta(s), \xi(s)) p(s) + \nabla_x F(p(s), \zeta(s), \xi(s)) \\
\dot{\zeta}(s) = \langle \nabla_p F(p(s), \zeta(s), \xi(s)), p(s) \rangle_{\mathbb{R}^n}.
\end{cases}
\]

with the prescribed conditions

\[ \begin{align*}
\zeta(0) &= \zeta(0) = \tilde{\xi}(y'), \\
p(0) &= p(0) = \tilde{\xi}(y'), \\
\xi(0) &= \xi(0) = g(y')
\end{align*} \]

and adequate \(\tilde{\xi}(\cdot)\), so that \(\tilde{\xi}(\xi') = q\) and for all \(y'\) the triple \((\tilde{\xi}(y'), g(y'), \begin{pmatrix} q_1 & \ldots & q_n \end{pmatrix} \begin{pmatrix} y' \\ 0 \end{pmatrix})\) is admissible, i.e.

\[
\begin{cases}
q_i(y') = \frac{\partial}{\partial x_i} g(y') \quad i \in \{1, \ldots, n-1\}, \\
F(\tilde{\xi}(y'), g(y'), \begin{pmatrix} y' \\ 0 \end{pmatrix}) = 0
\end{cases}
\]

holds for all \(y' \in \Omega'\) close to \(\xi'\). The existence of such a function \(\tilde{\xi}(\cdot)\) is guaranteed by the following lemma:

**Lemma 7.2** (Noncharacteristic prescribed data). Let \(\Omega', F\) be as above and \((q, \zeta, \begin{pmatrix} \xi' \\ 0 \end{pmatrix})\) be admissible. Then there exists a unique solution \(\tilde{\xi} : B_{\varepsilon}(\xi') \cap \Omega' \to \mathbb{R}^n\) with \(\tilde{\xi}(\cdot) = \begin{pmatrix} q_1(\cdot) \\ \vdots \\ q_n(\cdot) \end{pmatrix}\)

\[
\begin{cases}
\tilde{\xi}(\xi') = q, \\
q_i(y') = \frac{\partial}{\partial x_i} g(y') \quad i \in \{1, \ldots, n-1\}, \\
F(\tilde{\xi}(y'), g(y'), \begin{pmatrix} y' \\ 0 \end{pmatrix}) = 0
\end{cases}
\]

for all \(y' \in B_{\varepsilon}(\xi') \cap \Omega'\), provided that

\[ \frac{\partial}{\partial p_n} F(q, \zeta, \begin{pmatrix} \xi' \\ 0 \end{pmatrix}) \neq 0. \] (7.3)

An admissible triple \((q, \zeta, \begin{pmatrix} \xi' \\ 0 \end{pmatrix})\) is said to be \textit{noncharacteristic} if (7.3) holds.
Proof. Consider the function $\overline{G} : \mathbb{R}^n \times \Omega' \to \mathbb{R}^n$ with $\overline{G}(\cdot) = \begin{pmatrix} G_1(\cdot) \\ \vdots \\ G_n(\cdot) \end{pmatrix}$ given by

$$G_i(p, y') = p_i - \frac{\partial}{\partial x_i} g(y') \quad i \in \{1, \ldots, n-1\},$$

$$G_n(p, y') = F(p, g(y'), \begin{pmatrix} y' \\ 0 \end{pmatrix}).$$

Then, $\overline{G}(q, \xi') = 0$, since $(q, \zeta, \begin{pmatrix} \xi' \\ 0 \end{pmatrix})$ is admissible, and moreover,

$$D_p \overline{G}(q, \xi') = \begin{pmatrix} 1 & \cdots & (n-1 \text{ times}) & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{\partial}{\partial p_1} F(q, \zeta, \begin{pmatrix} \xi' \\ 0 \end{pmatrix}) & \cdots & \cdots & \frac{\partial}{\partial p_n} F(q, \zeta, \begin{pmatrix} \xi' \\ 0 \end{pmatrix}) \end{pmatrix} \Rightarrow \det D_p \overline{G}(q, \xi') = \frac{\partial}{\partial p_n} F(q, \zeta, \begin{pmatrix} \xi' \\ 0 \end{pmatrix}) \neq 0.$$

Thus, it follows from the implicit function theorem that the identity $\overline{G}(p, y') = 0$ can be uniquely expressed as $p = \overline{\eta}(y')$ with an adequate function $\overline{\eta}(\cdot)$ defined on $B_{\varepsilon}(\xi') \cap \Omega'$ a neighborhood of $\xi'$.

\textit{Lemma 7.3} (Local invertibility). Let $\Omega', F$ be as above and $(q, \zeta, \begin{pmatrix} \xi' \\ 0 \end{pmatrix})$ be noncharacteristic. Then there exists $t > 0$, a neighborhood $W(\xi') \subseteq \Omega' \subseteq \mathbb{R}^{n-1}$, a neighborhood $V(\begin{pmatrix} \xi' \\ 0 \end{pmatrix}) \subseteq \Omega' \times \mathbb{R} \subseteq \mathbb{R}^n$ such that for any $x \in V$, there exists a unique projected characteristic $\varpi : (-t, t) \to \Omega$, $\varpi \in (-t, t)$, $\overline{y} \in W$ such that:

$$x = \varpi(\overline{y}) \quad \text{with} \quad \varpi(0) = \overline{y}.$$

\textit{Proof.} By assumption: $(q, \zeta, \begin{pmatrix} \xi' \\ 0 \end{pmatrix})$ is noncharacteristic, so that by Lemma 7.2 that

$$\begin{cases} \overline{\eta}(\xi') = q, \\ q_i(y') = \frac{\partial}{\partial x_i} g(y') \quad i \in \{1, \ldots, n-1\}, \\ F(\overline{\eta}(y'), g(y'), \begin{pmatrix} y' \\ 0 \end{pmatrix}) = 0 \end{cases}$$

is uniquely solvable for $\overline{\eta}(\cdot)$. Thus, the compatibility conditions are satisfied in a neighborhood of $\xi'$ and it follows form the theory of ode’s that for any given point $y' \in \Omega'$ close to $\xi'$ the ode system of characteristics

$$\begin{cases} \dot{\varpi}(s) = \nabla_p F(\varpi(s), \varpi(s), \varpi(s)) \\ \dot{p}(s) = -\frac{\partial}{\partial \varpi} F(\varpi(s), \varpi(s), \varpi(s)) p(s) + \nabla_x F(\varpi(s), \varpi(s), \varpi(s)), \\ \dot{z}(s) = \langle \nabla_p F(\varpi(s), \varpi(s), \varpi(s)), p(s) \rangle_{\mathbb{R}^n}, \end{cases}$$

with the prescribed conditions

$$\varpi(0) = \begin{pmatrix} y' \\ 0 \end{pmatrix}, \quad p(0) = \overline{\eta}(y'), \quad \varpi(0) = g(y')$$

is solvable. For the projected characteristics we thus obtain a $C^1$-map:

$$\mathbb{R}^n \supset \Omega' \supset \mathbb{R} \ni \begin{pmatrix} y' \\ s \end{pmatrix} \mapsto \varpi \in \mathbb{R}^n.$$
Goal: show that this map is invertible around \( \xi' \). This will be achieved using the inverse function theorem: For \( \xi' \in \Omega' \) we have that the corresponding projected characteristic \( \varpi(\cdot) \) satisfies: \( \varpi(0) = \begin{pmatrix} \xi' \\ 0 \end{pmatrix} \). Moreover, for all \( y' \) in a neighborhood of \( \xi' \) the corresponding projected characteristic satisfies \( \varpi(0) = \begin{pmatrix} y' \\ 0 \end{pmatrix} \). Hence,

\[
D_{y'} \varpi(0) = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & \cdots & 0
\end{pmatrix}
\]

and \( \frac{\partial}{\partial s} \varpi(0) \equiv \nabla_{\varpi} F(\varpi(0), \varpi(0), \varpi(0)) \). \( \Box \)

By the noncharacteristic assumption. Hence, we can apply the inverse function theorem to conclude that the \( C^1 \)-map \( y' \mapsto \varpi \) is locally a \( C^1 \)-diffeomorphism.

By Lemma 7.3 we have a \( C^1 \)-diffeomorphism

\[
V \ni x \mapsto \begin{pmatrix} y' \\ s \end{pmatrix} \in W \times (-t, t).
\] (7.4)

Now, for fixed \( x \in V \) consider the corresponding projected characteristic through \( y'(x) \). Moreover, the characteristic ODE's are solvable with the corresponding prescribed data. We set

\[
\begin{align*}
u(x) & := \varpi(s(x)), \\
p(x) & := \varpi(s(x)) \quad \text{with } s(\cdot) \text{ from above.}
\end{align*}
\]

**Theorem 7.4 (Local existence).** With the definitions from Lemma 7.3 we obtain: The above solution \( u \) is a \( C^2 \)-solution of the PDE

\[
F(\nabla u, u, x) \equiv 0 \quad \text{for } x \in V,
\]

with prescribed data on \( \Omega' \times \{0\} \):

\[
u \begin{pmatrix} x' \\ 0 \end{pmatrix} = g(x') \quad \forall \begin{pmatrix} x' \\ 0 \end{pmatrix} \in (\Omega' \times \{0\}) \cap V.
\]

**Proof.** 1. step: Let \( y' \in W \). As in the proof of Lemma 7.3 we solve the characteristics ODE system and denote the corresponding solutions by \( \varpi(\cdot), \varpi(\cdot), \varpi(\cdot) \). We claim:

\[
F(\varpi(s), \varpi(s), \varpi(s)) = 0 \quad \forall s \in (-t, t).
\]

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We have
\[
\frac{\partial}{\partial s} \tilde{f}(s) = \langle \nabla_p F(p(s), \pi(s), \dot{\pi}(s)), \dot{\pi}(s) \rangle_{\mathbb{R}^n} + \frac{\partial}{\partial z} F(p(s), \pi(s), \dot{\pi}(s)) \dot{\pi}(s) + \langle \nabla_x F(p(s), \pi(s), \dot{\pi}(s)), \dot{\pi}(s) \rangle_{\mathbb{R}^n} \]
\[
= \langle \nabla_p F(\cdots), - \frac{\partial}{\partial z} F(\cdots) \rangle_{\mathbb{R}^n} + \frac{\partial}{\partial z} F(\cdots) \rangle_{\mathbb{R}^n} + \langle \nabla_x F(\cdots), \dot{\pi}(s) \rangle_{\mathbb{R}^n} = 0 \quad \forall s \quad \Rightarrow \quad \tilde{f} \equiv \text{const.}
\]

Since \( \tilde{f}(0) = F(p(0), \pi(0), \dot{\pi}(0)) = F(\eta(y'), g(y'), \begin{pmatrix} y' \\ 0 \end{pmatrix}) \) \( \equiv 7.2 \) we conclude the 1. step: \( \tilde{f} \equiv 0 \).

2. step: Let \( x \in V \) be arbitrary and \( \nabla_x \text{-diffeo} \) obtain unique \( y'(x) \in W, s(x) \in (-t, t) \) and denote the unique solutions of the corresponding ode's by \( \pi(\cdot), \pi'(\cdot), \ddot{\pi}(\cdot) \). By construction we have \( x = \pi(s(x)) \) and set
\[
u(x) := \pi(s(x)), \quad p(x) := p(s(x)) \]

By the 1. step we have:
\[
F(p(x), u(x), x) = F(p(s(x)), \pi(s(x)), \dot{\pi}(s(x))) = 0
\]
and for the conclusion of the theorem it suffices to show, that with the above choices it holds \([p(x) = \nabla u(x)] \) for all \( x \in V \).

To this end, note that
\[
\frac{\partial}{\partial s} \pi(s) = \dot{\pi}(s) \equiv \langle \nabla_p F(\cdots), p(s) \rangle_{\mathbb{R}^n} \equiv \langle \ddot{\pi}(s), p(s) \rangle \tag{7.5a}
\]
and
\[
D_y \pi(s) = \frac{\partial}{\partial s} [\pi(s)]' D_y \pi(s) \tag{7.5b}
\]

In order to show the last equality (7.5b) consider the function \( r(\cdot) \) given by
\[
[r(s)]' := D_y \pi(s) - [\pi(s)]' D_y \pi(s)
\]
with values in \( \mathbb{R}^{n-1} \). We have
\[
[r(0)]' = D_y \pi(0) - [\pi(0)]' D_y \pi(0) = D_y g(y') - [\pi(y')]'
\]
\[
\begin{bmatrix}
1 \\
0 \\
0, \ldots, 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
\text{comp cond}
\end{bmatrix}
\begin{bmatrix}
q_1(y') \\
\vdots \\
q_{n-1}(y')
\end{bmatrix}
\begin{bmatrix}
q_1(y') \\
\vdots \\
q_{n-1}(y')
\end{bmatrix} = 0.
\]

Furthermore,
\[
\frac{d}{ds} [r(s)]' = D_y \pi(s) - [\pi(s)]' D_y \pi(s) - [\pi(s)]' D_y \pi(s)
\]
\[
\overset{\text{Schwarz}}{=} D_y \pi(s) \nabla_p F(\cdots), p(s) \rangle_{\mathbb{R}^n} + \frac{\partial}{\partial z} F(\cdots) [\pi(s)]' D_y \pi(s) + [\nabla_x F(\cdots)]' D_y \pi(s) - [\pi(s)]' D_y \nabla_p F(\cdots) \pi(s)
\]
\[
= [\pi(s)]' D_y \pi(s) \nabla_p F(\cdots) + [\nabla_p F(\cdots)]' D_y p(s) + \frac{\partial}{\partial z} F(\cdots) \{D_y \pi(s) - [r(s)]' \} + D_x F(\cdots) D_y \pi(s)
\]
\[
- [\pi(s)]' D_y \nabla_p F(\cdots)
\]
\[
= - \frac{\partial}{\partial z} F(\cdots) [r(s)]' + D_y F(\cdots) D_y \ddot{\pi}(s) + [\nabla_x F(\cdots)]' D_y \pi(s) + D_x F(\cdots) D_y \pi(s)
\]
\[
\overset{\text{chain rule}}{=} - \frac{\partial}{\partial z} F(\cdots) [r(s)]' + D_y \left[F(p(s), \pi(s), \dot{\pi}(s)) \right] = - \frac{\partial}{\partial z} F(\cdots) [r(s)]'.
\]

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Summarizing, the function \( r(\cdot) \) solves the linear ODE (system):
\[
\dot{r}(s) = -\frac{\partial}{\partial z} F(\cdots) r(s) \quad \text{with} \quad r(0) = 0 \implies r \equiv 0.
\]

Having established (7.5b) we conclude:
\[
D_x u(x) = D_x \tau(s(x)) \quad \text{chain rule} \quad \frac{\partial}{\partial s} \tau(s(x)) D_x s(x) + D_y \tau(s(x)) D_x y'(x)
\]

(7.5a)
(7.5b)
\[
\bigg\{ \frac{\partial}{\partial s} \tau(s(x)), \big[ \tau(s(x)) \big] \bigg\}_R \bigg\{ D_x s(x) + \big[ \tau(s(x)) \big]^T D_y \tau(s(x)) D_x y'(x) \bigg\} \quad \text{chain rule} \quad \bigg[ \tau(s(x)) \bigg]^T D_x \bigg[ \tau(s(x)) \bigg] = 0
\]
\[
= \bigg[ \tau(s(x)) \bigg]^T D_x x = \bigg[ \tau(s(x)) \bigg]^T = \bigg[ p(x) \bigg]^T
\]
as desired. \( \square \)

So far, we focused on the case when the data is prescribed on a hyperplane (flat space). If the data is prescribed on a nonflat \( C^1 \)-manifold (hypersurface) \( \implies \) flatten it:

Let \( \Omega \subseteq \mathbb{R}^n \) be open and \( \Gamma \) be an \((n-1)\)-dimensional \( C^1 \)-manifold, meaning that for all \( \xi \in \Gamma \) there exists a radius \( r > 0 \), an open set \( \Omega' \subseteq \mathbb{R}^{n-1} \) and a \( C^1 \)-function \( \gamma : \Omega' \to \mathbb{R} \) such that
\[
\Gamma \cap B_r(\xi) = \text{graph}(\gamma) = \left\{ \left( \begin{array}{c} y' \\ \gamma(y') \end{array} \right) \mid y' \in \Omega' \right\}.
\]
Consider the function \( \Phi : B_r(\xi) \to \mathbb{R}^n \) given by
\[
\Phi(x) = \Phi(x') = \begin{pmatrix} x' \\ x_n - \gamma(x') \end{pmatrix}.
\]
This map \( \Phi \) flattens \( \Gamma \cap B_r(\xi) \), indeed, we have
\[
\Phi(\Gamma \cap B_r(\xi)) = \Omega' \times \{0\}.
\]
Note, that \( \Phi \) is a \( C^1 \)-diffeomorphism with inverse function given by
\[
\Phi^{-1}(y) = \begin{pmatrix} y' \\ y_n + \gamma(y') \end{pmatrix}.
\]
Hence,
\[
u(x) = u(\Phi^{-1}(y)) =: v(y) = v(\Phi(x)) \quad \Rightarrow \quad D_x u(x) = (D_y v)(\Phi(x)) \cdot D_x \Phi(x)
\]
and our PDE reads:
\[
0 \equiv F(\nabla u(x), u(x), x) = F([D_x \Phi(x)]^T (\nabla_y v)(\Phi(x)), v(\Phi(x)), x) \quad \text{with admissible } G(\cdot, \cdot, \cdot). \quad \text{The data was prescribed on } \Gamma: u(x) = g(x) \quad \text{for all } x \in \Gamma. \quad \text{Thus, for all } x = \Phi(y) \in \Gamma \cap B_r(\xi) \quad \text{we obtain:}
\]
\[
v(y) = u(\Phi^{-1}(y)) = g(\Phi^{-1}(y)) := \bar{g}(y') \quad \text{with } y = \begin{pmatrix} y' \\ 0 \end{pmatrix} \in \Omega' \times \{0\}.
\]
Since this new data \( \bar{g}(\cdot) \) is prescribed on a flat space and the problem has the same form we can perform the arguments for the flat situation (Theorem 7.4).
Question: How does the noncharacteristic condition look like in the nonflat case?

Recall that in the flat case the noncharacteristic condition for an admissible triple \((q, \zeta, \ell_0')\) reads:

\[
0 \neq \frac{\partial}{\partial p_n} G(q, \zeta, \ell_0') = \langle \nabla_p G(q, \zeta, \ell_0'), e_n \rangle_{\mathbb{R}^n}
\]

Hence, the noncharacteristic condition in the nonflat case becomes:

\[
\langle \nabla_p F(e_q, e_\zeta, e_\ell), \nu(e_\ell) \rangle_{\mathbb{R}^n} \neq 0
\]

whereby \(\nu(\tilde{\ell})\) denotes a normal to \(\Gamma\) at \(\tilde{\ell}\).

**7.2 Semilinear pde’s of 2nd order**

Let \(\Omega \subseteq \mathbb{R}^n\) be open, consider the semilinear pde

\[
\sum_{|\alpha|=2} a_\alpha(x) D^\alpha u(x) + \vec{F}(\nabla u(x), u(x), x) \equiv 0 \quad \text{in } \Omega,
\]

here we rewrite

\[
\sum_{|\alpha|=2} a_\alpha(x) D^\alpha u(x) = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} u(x) A=(a_{ij})_{i,j=1}^n \langle A(x), D\nabla u(x) \rangle_{\mathbb{R}^{n \times n}} = \text{tr} \left( A(x) D\nabla u(x) \right).
\]

Furthermore, the data is prescribed on an \((n-1)\)-dimensional \(C^1\)-manifold \(\Gamma\):

\[
\begin{cases}
    u(x) |_{\Gamma} = g(x), \\
    \frac{\partial u}{\partial \nu}(x) |_{\Gamma} = h(x),
\end{cases}
\]

whereby \(\nu(\cdot)\) denotes a unit normal field on \(\Gamma\) and since a solution is given in the oben set \(\Omega\) the normal derivative can also be seen as

\[
\frac{\partial u}{\partial \nu}(x) = D_\nu u(x) = \langle \nabla u(x), \nu(x) \rangle_{\mathbb{R}^n} \quad \text{for } x \in \Gamma.
\]

Let us moreover denote by \(\tau_1(\cdot), \ldots, \tau_{n-1}(\cdot)\) the \((n-1)\) unit tangent vector fields, so that \(\{\tau_1(\cdot), \ldots, \tau_{n-1}(\cdot), \nu(\cdot)\}\) is an \(n\)-frame along \(\Gamma\), meaning that at each point \(x \in \Gamma\) the vectors \(\tau_1(x), \ldots, \tau_{n-1}(x)\) span the tangent space at \(x\) and \(\{\tau_1(x), \ldots, \tau_{n-1}(x), \nu(x)\}\) forms an (oriented orthonormal) basis of \(\mathbb{R}^n\). Since \(g(\cdot)\) and \(h(\cdot)\) are prescribed on \(\Gamma\) we can differentiate them in the tangent directions and obtain for all \(x \in \Gamma\):

\[
D_\tau j g(x) = D_\tau j u(x) = \langle \nabla u(x), \tau_j(x) \rangle_{\mathbb{R}^n} \quad \text{for } j = 1, \ldots, n-1,
\]
D_x D_{\tau_j} g(x) = D_{\tau_i} D_{\tau_j} u(x) = [\tau_i(x)]^T D \nabla u(x) \tau_j(x) = \{[\tau_i(x)]^T D \nabla u(x) \tau_j(x)\}^T

= [\tau_j(x)]^T D \nabla u(x) \tau_i(x) = D_{\tau_i} D_{\tau_j} u(x) = D_{\tau_j} D_{\tau_i} g(x) \quad \text{for } i, j = 1, \ldots, n - 1,

D_{\tau_j} h(x) = D_{\tau_j} D_{\tau_i} u(x) = [\tau_j(x)]^T D \nabla u(x) \nu(x) \quad \text{for } j = 1, \ldots, n - 1.

Recall that for fixed \( x \in \Gamma \) the vectors \{\tau_1(x), \ldots, \tau_{n-1}(x), \nu(x)\} form an (oriented orthonormal) basis of \( \mathbb{R}^n \). With the previous calculations we can recover from the given Cauchy data already the values of the solution \( u \) and of its gradient \( \nabla u \) completely in this point \( x \in \Gamma \):

\[ u(x) = g(x), \]

\[ \nabla u(x) = \sum_{j=1}^{n-1} \langle \nabla u(x), \tau_j(x) \rangle_{\mathbb{R}^n} \tau_j(x) + \langle \nabla u(x), \nu(x) \rangle_{\mathbb{R}^n} \nu(x) = \sum_{j=1}^{n-1} D_{\tau_j} g(x) \tau_j(x) + h(x) \nu(x). \]

For the entries of the Hessian we obtain in this basis:

\[ D \nabla u(x) = \sum_{i,j=1}^{n-1} \left( [\tau_i(x)]^T D \nabla u(x) \tau_j(x) \right) \tau_j(x) \tau_i(x) + \sum_{j=1}^{n-1} \left( [\tau_j(x)]^T D \nabla u(x) \nu(x) \right) \tau_j(x) \nu(x) \]

\[ + \left( [\nu(x)]^T D \nabla u(x) \nu(x) \right) \nu(x) \nu(x) \]

\[ = \sum_{i,j=1}^{n-1} D_{\tau_i} D_{\tau_j} g(x) \tau_j(x) \tau_i(x) + \sum_{j=1}^{n-1} D_{\tau_j} h(x) \tau_j(x) \nu(x) + \left( [\nu(x)]^T D \nabla u(x) \nu(x) \right) \nu(x) \nu(x). \]

So that, for the complete recovery of the Hessian we need to determine the missing term \([\nu(x)]^T D \nabla u(x) \nu(x)\). This is achieved from our pde considered at \( x \in \Gamma \):

\[ -\vec{F}(\nabla u(x), u(x), x) = \langle A(x), D \nabla u(x) \rangle_{\mathbb{R}^n} = \sum_{i,j=1}^{n-1} D_{\tau_i} D_{\tau_j} g(x) \langle A(x), \tau_j(x) \tau_i(x) \rangle \]

\[ + \sum_{j=1}^{n-1} D_{\tau_j} h(x) \langle A(x), \tau_j(x) \nu(x) \rangle + \left( [\nu(x)]^T D \nabla u(x) \nu(x) \right) \langle A(x), \nu(x) \nu(x) \rangle, \]

meaning that we can recover the missing value \([\nu(x)]^T D \nabla u(x) \nu(x)\) provided that \(0 \neq \langle A(x), \nu(x) \nu(x) \rangle \in \mathbb{R}^{n \times n} = \nu(x)^T A(x) \nu(x) \) and we obtain as

**noncharacteristic condition:** \([\nu(x)]^T A(x) \nu(x) \neq 0.\)

In this case all the entries of the Hessian \( D \nabla u(x) \) can be reconstructed completely for \( x \in \Gamma \). In the same manner consider higher order directional derivatives of the Cauchy data and differentiate the given pde in order to recover higher order derivatives of the solution. This is enough to reconstruct \( u \) by a local Taylor expansion and show that this series converges in a neighborhood of \( x \). Hence, assuming enough regularity of the given data we obtain a local analytic solution. This is one version of the **Cauchy-Kowalewskaja theorem** and for the details we refer to the original paper


**Remark 7.5.** If our semilinear pde \( \langle A(x), D \nabla u(x) \rangle + \vec{F}(\nabla u(x), u(x), x) \equiv 0 \) is elliptic, i.e. all eigenvalues of \( A(x) \) are positive or all eigenvalues are negative, so that for all \( x \in \Gamma \):

\[ v^T A(x) v > 0 \quad \forall \ v \in \mathbb{R}^n \setminus \{0\} \quad \text{or} \quad v^T A(x) v < 0 \quad \forall \ v \in \mathbb{R}^n \setminus \{0\} \]

then the noncharacteristic condition is fulfilled everywhere and we obtain directly local solvability.

**Examples:**

1) With the choice \( A(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E_n \) and \( \vec{F}(p, z, x) = f(x) \) we obtain Poisson’s equation: \( \Delta u(x) + f(x) \equiv 0. \)

Since the eigenvalues of \( A(x) \) are everywhere \( \lambda_1 = \ldots = \lambda_n = 1 > 0 \), this is an elliptic equation.
2) With the choice $A(x) = \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix}$ and $\vec{F}(p, z, x) = f(x)$ we obtain the inhomogeneous wave equation in $(n-1)$-spatial dimensions:

$$-\frac{\partial^2}{\partial x_1^2} u(x) + \frac{\partial^2}{\partial x_2^2} u(x) + \cdots + \frac{\partial^2}{\partial x_n^2} u(x) + f(x) \equiv 0 \quad \text{with } t \equiv x_1.$$ 

Since the eigenvalues of $A(x)$ are everywhere $\lambda_1 = -1 < 0$ and $\lambda_2 = \ldots = \lambda_n = 1 > 0$, this is a hyperbolic equation.

3) With the choice $A(x) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix}$ and $\vec{F}(p, z, x) = -p_1 + f(x)$ we obtain the inhomogeneous heat equation in $(n-1)$-spatial dimensions:

$$\frac{\partial^2}{\partial x_1^2} u(x) + \cdots + \frac{\partial^2}{\partial x_n^2} u(x) - \frac{\partial}{\partial x_1} u(x) + f(x) \equiv 0 \quad \text{with } t \equiv x_1.$$ 

Since the eigenvalues of $A(x)$ are everywhere $\lambda_1 = 0$ and $\lambda_2 = \ldots = \lambda_n = 1 > 0$, this is a parabolic equation.

4) Note, that the type even of a linear equation might be different in different regions: In $n = 2$ dimensions consider $A(x) = \begin{pmatrix} x_2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\vec{F}(p, z, x) \equiv 0$, so that we obtain the Tricomi equation:

$$x_2 u_{x_1, x_1} + u_{x_2 x_2} \equiv 0.$$ 

Since the eigenvalues of $A(x)$ are $\lambda_1 = x_2$ and $\lambda_2 = 1$ we have:

- if $x_2 < 0$ then the equation is hyperbolic,
- if $x_2 = 0$ then the equation is parabolic,
- if $x_2 > 0$ then the equation is elliptic.

Let us now revisit quasilinear pde’s of 2nd order in two dimensions:

$$a_{11} \left( \nabla u \left( \frac{x_1}{x_2}, u \left( \frac{x_1}{x_2}, \frac{x_2}{x_1} \right) \right) \right) + a_{12} \left( \nabla u \left( \frac{x_1}{x_2}, u \left( \frac{x_1}{x_2}, \frac{x_2}{x_1} \right) \right) \right) \frac{\partial^2}{\partial x_1 \partial x_2} u \left( \frac{x_1}{x_2}, \frac{x_2}{x_1} \right) + a_{21} \left( \nabla u \left( \frac{x_1}{x_2}, u \left( \frac{x_1}{x_2}, \frac{x_2}{x_1} \right) \right) \right) \frac{\partial^2}{\partial x_2 \partial x_1} u \left( \frac{x_1}{x_2}, \frac{x_2}{x_1} \right) + a_{22} \left( \nabla u \left( \frac{x_1}{x_2}, u \left( \frac{x_1}{x_2}, \frac{x_2}{x_1} \right) \right) \right) \frac{\partial^2}{\partial x_2^2} u \left( \frac{x_1}{x_2}, \frac{x_2}{x_1} \right)$$

$$+ \vec{F} \left( \nabla u \left( \frac{x_1}{x_2}, u \left( \frac{x_1}{x_2}, \frac{x_2}{x_1} \right) \right) \right) = 0.$$ 

Renaming the coordinates $(x_1, x_2)$ to $(x, y)$ we can rewrite this equation to

$$0 = \begin{pmatrix} a_{xx} u_{xx} & a_{xy} u_{xy} & a_{yy} u_{yy} \\ u_{xx} & u_{xy} & u_{yy} \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} + 2 b \langle \ldots \rangle u_{xy} \begin{pmatrix} x \\ y \end{pmatrix} + c \langle \ldots \rangle u_{yy} \begin{pmatrix} x \\ y \end{pmatrix} + \vec{F} \langle \ldots \rangle$$

$$= \begin{pmatrix} \langle \begin{pmatrix} a(x, y) b(x, y) \cdots \end{pmatrix} \langle \begin{pmatrix} u_{xx} u_{xy} u_{yy} \end{pmatrix} \rangle_R \end{pmatrix}_{2 \times 2} + \vec{F} \langle \ldots \rangle \quad \text{in } \Omega \text{ (open)} \subseteq \mathbb{R}^2$$
with data prescribed on a $C^3$-curve $\Gamma$ inside $\Omega$:

$$
\Gamma = \left\{ \begin{pmatrix} x(s) \\ y(s) \end{pmatrix}, s \in I (\text{interval} \subseteq \mathbb{R} ) \right\}
$$

whereby $[\dot{x}(s)]^2 + [\dot{y}(s)]^2 \neq 0$ for all $s \in I$ meaning, that the tangent exists everywhere on $\Gamma$. The data is given by

$$
u \begin{pmatrix} u(x(s), y(s)) \\ u_x(x(s), y(s)) \\ u_y(x(s), y(s)) \end{pmatrix} = \begin{pmatrix} g(s) \\ h_1(s) \\ h_2(s) \end{pmatrix}, \quad (7.6a)$$

However, not arbitrary functions can be chosen here, in fact the following necessary condition should hold: Differentiating $(7.6a)$ with respect to $s$ we obtain:

$$
\dot{g}(s) = \frac{d}{ds} u \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = u_x \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} \dot{x}(s) + u_y \begin{pmatrix} x(s) \\ y(s) \end{pmatrix} \dot{y}(s) = \begin{pmatrix} h_1(s) \\ h_2(s) \end{pmatrix} \begin{pmatrix} \dot{x}(s) \\ \dot{y}(s) \end{pmatrix} \in \mathbb{R}^2.
$$

If this compatibility condition is satisfied the quintupel $(x(s), y(s), g(s), h_1(s), h_2(s))$ is called initial strip of first order, and no more then two of the functions $g, h_1, h_2$ can be chosen arbitrary. Instead the prescribed data on $\Gamma$ might be the values of $u$ and of its normal derivative:

$$
u \begin{pmatrix} u(x(s), y(s)) \\ \frac{\partial}{\partial y} u(x(s), y(s)) \end{pmatrix} = \begin{pmatrix} g(s) \\ \frac{1}{\sqrt{[\dot{x}(s)]^2 + [\dot{y}(s)]^2}} \begin{pmatrix} u_x(\cdots) \\ u_y(\cdots) \end{pmatrix} \begin{pmatrix} -\dot{y}(s) \\ \dot{x}(s) \end{pmatrix} \end{pmatrix}, \quad (7.6b)
$$

whereby $\frac{1}{\sqrt{[\dot{x}(s)]^2 + [\dot{y}(s)]^2}} \begin{pmatrix} -\dot{y}(s) \\ \dot{x}(s) \end{pmatrix}$ denotes the unit normal field along $\Gamma$.

In order to obtain compatibility conditions for higher order derivatives we differentiate $(7.6b)$ and $(7.6c)$:

$$
\dot{h}_1(s) = \frac{d}{ds} u_x \begin{pmatrix} -\dot{y}(s) \\ \dot{x}(s) \end{pmatrix} = u_{xx}(\cdots) \dot{x}(s) + u_{xy}(\cdots) \dot{y}(s),
$$

$$
\dot{h}_2(s) = \frac{d}{ds} u_y \begin{pmatrix} -\dot{y}(s) \\ \dot{x}(s) \end{pmatrix} = u_{xy}(\cdots) \dot{x}(s) + u_{yy}(\cdots) \dot{y}(s).
$$

Hence, a solution should along $\Gamma$ satisfy

$$
\begin{pmatrix} a(\cdots) & 2b(\cdots) & c(\cdots) \\ \dot{x}(s) & \dot{y}(s) & 0 \\ 0 & \dot{x}(s) & \dot{y}(s) \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} -\tilde{F}(\cdots) \\ \dot{h}_1(s) \\ \dot{h}_2(s) \end{pmatrix}.
$$

This system is uniquely solvable provided that the determinant of the matrix is distinct from zero:

$$
0 \neq \det(\cdot) = a(\cdots)[\dot{y}(s)]^2 - 2b(\cdots)\dot{x}(s)\dot{y}(s) + c(\cdots)[\dot{x}(s)]^2 = (\dot{y}(s) \dot{x}(s)) \begin{pmatrix} a(\cdots) & b(\cdots) & c(\cdots) \\ \dot{x}(s) & \dot{y}(s) & 0 \\ 0 & \dot{x}(s) & \dot{y}(s) \end{pmatrix} = (\dot{x}(s))^2 + (\dot{y}(s))^2 \left[ \nu(s) \right]^T A(\cdots) \nu(s) \iff \left[ \nu(s) \right]^T A(\cdots) \nu(s) \neq 0,
$$

the well-known condition. Thus, $\Gamma$ is characteristic (with respect to the prescribed data) if $\det(\cdot) = 0$ along $\Gamma$, noncharacteristic (=not characteristic) if $\det(\cdot) \neq 0$ along $\Gamma$. In the latter case, the Cauchy data already determines uniquely all second derivatives of $u$ along $\Gamma$. As before consider then higher order derivatives in order to reconstruct the solution locally by Taylor series (Cauchy–Kowalewskaja).
Note: If the Cauchy data is prescribed on a characteristic curve, then in general a solution cannot be reconstructed, to this end need additional conditions.

We now turn our attention to the construction of characteristic curves:

Let locally \( \Gamma \) be described by a graph of a function \( \gamma \), w.l.o.g. say \( y = \gamma(x) \) so that \( y(s) = \gamma(x(s)) \) and \( \dot{y}(s) = \gamma'(x(s))\dot{x}(s) \) and \( 0 \neq [\dot{x}(s)]^2 + [\dot{y}(s)]^2 = [\dot{x}(s)]^2(1 + [\gamma'(x(s))]^2) \) iff \( \dot{x}(s) \neq 0 \). Hence, the characteristic condition becomes

\[
a(s) [\gamma'(x(s))]^2 [\dot{x}(s)]^2 - 2b(s) \gamma'(x(s)) [\dot{x}(s)]^2 + c(s) [\dot{x}(s)]^2 = 0
\]

This last equation is on one for \( \gamma \) provided that the coefficients \( a(s), b(s), c(s) \) are known functions of \( s \). This is the case when either a particular solution \( u \) is considered, or we are in the semilinear case, i.e. study an equation of type:

\[
a(x, y) u_{xx} + 2b(x, y) u_{xy} + c(x, y) u_{yy} + \tilde{F}(u, u_x, u_y, u, x, y) \equiv 0.
\]

In the latter case the characteristic condition becomes

\[
a(x, \gamma(x)) [\gamma'(x)]^2 - 2b(x, \gamma(x)) \gamma'(x) + c(x, \gamma(x)) = 0
\]

an ODE for \( \gamma(\cdot) \).

- If \( a(x, \gamma(x)) \neq 0 \) then

\[
\gamma'(x) = \frac{b(x, \gamma(x)) \pm \sqrt{[b(x, \gamma(x))]^2 - a(x, \gamma(x)) c(x, \gamma(x))}}{a(x, \gamma(x))}
\]

provided that the discriminant \( b^2 - ac \geq 0 \) for the argument \((x, \gamma(x))\).

(If \( b^2 - ac < 0 \), then no characteristic curve exists.)

- If \( a(x, \gamma(x)) = 0 \) and \( c(x, \gamma(x)) \neq 0 \) interchange the roles of \( x \) and \( y \) and look locally at a parametrization \( x = \psi(y) \).

Then \( \dot{x}(s) = \psi'(y(s))\dot{y}(s) \) and dividing the corresponding characteristic condition by \([\dot{y}(s)]^2\) we obtain

\[
a(\psi(y), y) - 2b(\psi(y), y) \psi'(y) + c(\psi(y), y)[\psi'(y)]^2 = 0.
\]

Hence,

\[
\psi'(y) = \frac{b(\psi(y), y) \pm \sqrt{[b(\psi(y), y)]^2 - a(\psi(y), y) c(\psi(y), y)}}{c(\psi(y), y)}
\]

provided that \( b^2 - ac \geq 0 \).

- If \( a(x, \gamma(x)) = c(x, \gamma(x)) = 0 \) and \( b(x, \gamma(x)) \neq 0 \) then we obtain the two characteristic curves given by

\[
\gamma'(x) \equiv 0 \quad \text{and} \quad \psi'(y) \equiv 0.
\]

Recall that a quasilinear PDE of 2nd order

\[
\left( \begin{array}{cc}
  a(s) & b(s) \\
  b(s) & c(s)
\end{array} \right) \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} = \tilde{F}(\cdot) \equiv 0 \quad \text{is called}
\]

elliptic iff \( ac - b^2 > 0 \) (both eigenvalues have the same sign),

parabolic iff \( ac - b^2 = 0 \) (one eigenvalue is zero),

hyperbolic iff \( ac - b^2 < 0 \) (eigenvalues have different sign).

Remark 7.6.

- In the elliptic case, no characteristic curves exist.

- In the parabolic case, there exists one characteristic curve through every point of \( \Omega \).

- In the hyperbolic case, there exist two characteristic curves through every point.
Examples:

1) With the choice \( \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \tilde{F}(p, z, \begin{pmatrix} x \\ y \end{pmatrix}) = f(x, y) \) we obtain Poisson’s equation: \( u_{xx} + u_{yy} + f(x, y) = 0. \)

An elliptic equation and \( b^2 - ac = -1 < 0 \) so that no characteristic curves exist.

2) With the choice \( \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \tilde{F}(p, z, \begin{pmatrix} x \\ y \end{pmatrix}) = f(x, y) \) we obtain the inhomogeneous wave equation in 1d:

\[ -u_{xx} + u_{yy} + f(x, y) = 0, \]

a hyperbolic equation and \( b^2 - ac = 1 > 0 \). The characteristic curves are given by

\[ \gamma'(x) = \frac{0 \pm \sqrt{1}}{-1} = \pm 1 \quad \Rightarrow \quad \gamma(x) = \pm x + c \]

and we obtain two families of characteristic curves. [See also our calculations in characteristic coordinates for the wave equation in 1D from Section 6.2, solutions can be written in the form \( u(x, y) = w_1(y - x) + w_2(y + x) \).]

3) With the choice \( \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \tilde{F}(p, z, \begin{pmatrix} x \\ y \end{pmatrix}) = -p_1 + f(x, y) \) we obtain the inhomogeneous heat equation in 1d:

\[ u_{yy} - u_x + f(x, y) = 0, \]

a parabolic equation and \( b^2 - ac = 0 \). The characteristic curves are given by:

\[ \psi'(y) = \frac{0 \pm \sqrt{0}}{1} = 0 \quad \Rightarrow \quad \psi(y) = \text{const.} \]

4) With the choice \( \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \) and \( \tilde{F}(p, z, \begin{pmatrix} x \\ y \end{pmatrix}) = 0 \) we obtain Tricomi’s equation:

\[ y u_{xx} + u_{yy} = 0. \]

We have \( ac - b^2 = y \) and

- if \( y > 0 \) the equation is elliptic and no characteristic curves exist,
- if \( y = 0 \) the equation is parabolic (no characteristic curves exists since we are only on the \( x \)-axis, cf. heat equation),
- if \( y < 0 \) the equation is hyperbolic and we obtain

\[ \gamma'(x) = \frac{0 \pm \sqrt{-\gamma(x)}}{\gamma(x)} = \pm(-\gamma(x))^{-1/2} \quad \Rightarrow \quad \gamma(x) = -\left(\frac{3}{2}\right)^{2/3} (x - c)^{2/3}. \]
Remark 7.7. Consider the quasilinear PDE

\[(1 + u_y^2)u_{xx} - 2u_x u_y + (1 + u_x^2)u_{yy} \equiv 0,\]

i.e. the minimal surface equation in 2d (cf. Exercise 9), and with \(a = 1 + u_y^2, b = -u_x u_y, c = 1 + u_x^2\) we have

\[ac - b^2 = (1 + u_y^2)(1 + u_x^2) - u_x^2 u_y^2 = 1 + u_x^2 + u_y^2 \geq 1 > 0,\]

meaning that the MSE is elliptic. Solutions of the MSE will be thoroughly studied in the lecture Minimal Surfaces held at KIT in the summer term 2024.

Also the "wrong minimal surface equation":

\[(1 + u_x^2)u_{xx} + 2u_x u_y + (1 + u_y^2)u_{yy} \equiv 0,\]

is elliptic, however, its solutions behave differently than the solutions of the MSE.

Remark 7.8. Recall that in the elliptic case, there are no characteristic curves and by the Cauchy-Kowalewskaja theorem we obtain locally a solution. However, a major drawback of the Cauchy-Kowalewskaja theorem is that it gives little control over the dependence of the solution on the Cauchy data, indeed, cf. Hadamard’s example in 2d:

\[
\begin{align*}
    u_{xx} + u_{yy} &\equiv 0 \\
    u(x,0) &= 0, \quad u_y(x,0) = k e^{-\sqrt{k} \sin(k x)} \quad \text{with } k \in \mathbb{N}.
\end{align*}
\]

The PDE is elliptic (Laplace eq.) and the solution is (Exercise):

\[u(x,y) = e^{-\sqrt{k}} \sin(kx) \sinh(ky).\] (7.7)

For \(k \to \infty\) we obtain for the Cauchy data: \(u(x,0) = 0\) and \(u_y(x,0) \xrightarrow{k \to \infty} 0\), so that the solution to the limiting case \(k = \infty\) is \(u \equiv 0\). But, for \(y \neq 0\)

\[e^{-\sqrt{k}} \sinh(ky) = e^{-\sqrt{k}} \frac{e^{ky} - e^{-ky}}{2} \xrightarrow{k \to \infty} \infty\]

meaning that the solution (7.7) oscillates more and more rapidly with growing \(k\).

This example shows that the solution may not depend continuously on the Cauchy data. The "correct" problems for elliptic equations are boundary value problems (cf. maximum principles etc.).

Remark 7.9 (The terms ‘elliptic’, ‘parabolic’, ‘hyperbolic’).

The ellipse, parabola and hyperbola can be seen as conic sections and written as quadratic polynomials of the form:

\[Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0.\]

- If \(AC - B^2 > 0\) we have an ellipse (simplest form: \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\); and circles are special cases of ellipses with \(B = 0\) and \(A = C\)).
- If \(AC - B^2 = 0\) we have a parabola (simplest form: \(x^2 = y\)).
- If \(AC - B^2 < 0\) we have a hyperbola (simplest form: \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0\)).

Figure 35: Indicated are cone sections.
8 Outlook

Recall Dirichlet’s principle (Theorem 5.2): Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain, \( g \in C^0(\partial \Omega; \mathbb{R}) \). Find \( u \in C^2(\Omega; \mathbb{R}) \cap C^0(\overline{\Omega}; \mathbb{R}) \):

\[
\begin{align*}
\Delta u &\equiv 0 \quad \text{in } \Omega, \\
u &\equiv g \quad \text{on } \partial \Omega.
\end{align*}
\]

\( \iff \) \( u \) minimizes the Dirichlet energy with prescribed boundary data, i.e., setting \( \mathcal{D}(v) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx \) and

\[
\mathcal{E}(g) := \{ v \in C^2(\Omega; \mathbb{R}) \cap C^0(\overline{\Omega}; \mathbb{R}) : v = g \text{ on } \partial \Omega \}
\]

it holds

\[
\mathcal{D}(u) = \min_{v \in \mathcal{E}(g)} \mathcal{D}(v).
\]

But why does an extrema even exist?

Example 8.1 (1 is the biggest natural number). We proof this claim by contradiction: Assume the contrary and let \( n > 1 \) be the biggest number, but then \( n^2 > n \) is even bigger.

Example 8.2 (Harmonic function with infinite Dirichlet energy). Recall the Laplacian in polar coordinates in 2d (Exercise 12a):

\[
\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}
\]

and for all \( k \in \mathbb{N} \) the functions \( r^k \cos(k\varphi) \) and \( r^k \sin(k\varphi) \) are harmonic (real and imaginary parts of \((x + iy)^k\)). On the closed unit disk \( \overline{B_1(0)} \subset \mathbb{R}^2 \) consider functions given in polar coordinates by the following series

\[
u(r, \varphi) = a_0 + \sum_{k=1}^{\infty} r^k (a_k \cos(k\varphi) + b_k \sin(k\varphi)), \quad r \in [0, 1], \quad \varphi \in [0, 2\pi).
\]

If \( \sum_{k=1}^{\infty} |a_k| + |b_k| < \infty \) the previous series converges uniformly on the closed unit disk \( \overline{B_1(0)} \) and its derivatives converge uniformly on any compact subset of the open unit disk. Hence, such functions

\[
u \in C^\infty(B_1(0); \mathbb{R}) \cap C^0(\overline{B_1(0)}; \mathbb{R})
\]

and are harmonic. More precisely, we have for \( r < 1 \):

\[
u_r = \sum_{k=1}^{\infty} k r^{k-1} (a_k \cos(k\varphi) + b_k \sin(k\varphi)) \quad \text{and} \quad \nu_\varphi = \sum_{k=1}^{\infty} k r^{k-1} (-a_k \sin(k\varphi) + b_k \cos(k\varphi))
\]

which converge uniformly for \( r < 1 \), so that for \( 0 < r < 1 \) we obtain

\[
u_r^2 + \left( \frac{1}{r} \nu_\varphi \right)^2 = \sum_{k=1}^{\infty} k^2 r^{2k-2} (a_k^2 + b_k^2)
\]

and for the Dirichlet integral:

\[
\mathcal{D}(u) = \frac{1}{2} \int_{B_1(0)} |\nabla u|^2 \, dx = \frac{1}{2} \int_0^{2\pi} \int_0^1 \left[ \nu_r^2 + \left( \frac{1}{r} \nu_\varphi \right)^2 \right] r \, dr \, d\varphi = \pi \sum_{k=1}^{\infty} k^2 r^{2k-2} (a_k^2 + b_k^2) \int_0^1 k^2 r^{2k-1} \, dr
\]

\[
= \frac{\pi}{2} \sum_{k=1}^{\infty} k (a_k^2 + b_k^2).
\]
Thus, taking $a_k = 0$ for all $k \in \mathbb{N}$ and $b_k = \begin{cases} \frac{1}{n^2} & \text{for } k = n! \\ 0 & \text{else} \end{cases}$ we consider

$$u(r, \varphi) = \sum_{n=1}^{\infty} \frac{r^n}{n^2} \sin(n! \varphi).$$

By the above calculations we have for this function: $u \in C^\infty(B_1(0); \mathbb{R}) \cap C^0(\overline{B_1(0)}; \mathbb{R})$ is harmonic but

$$\mathcal{D}(u) = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{n!}{n^2} = +\infty.$$

In fact, every function $v \in C^\infty(B_1(0); \mathbb{R}) \cap C^0(\overline{B_1(0)}; \mathbb{R})$ that coincides with this $u$ along $\partial B_1(0)$ has infinite Dirichlet energy.

**Example 8.3** (Weierstraß critique on Dirichlet’s principle). Consider the functional $\mathcal{F}(u) := \int_{-1}^{1} x^2[u'(x)]^2 \, dx$ and minimize it among the class

$$\mathcal{C} := \{ v \in C^1((-1, 1); \mathbb{R}) : v(1) = 1, v(-1) = -1 \}.$$

We have $\mathcal{F}(u) \geq 0$ so that $\inf_{v \in \mathcal{C}} \mathcal{F}(v) \geq 0$. For the sequence $u_\varepsilon(x) := \arctan \left( \frac{\varepsilon}{x} \right)$

we obtain $u_\varepsilon \in \mathcal{C}$ for all $\varepsilon > 0$ and by direct calculations (Exercise) $\mathcal{F}(u_\varepsilon) \xrightarrow{\varepsilon \to 0} 0$

meaning that $\inf_{v \in \mathcal{C}} \mathcal{F}(v) = 0$.

But, there is no function $u \in \mathcal{C}$ such that $\mathcal{F}(u) = 0$, since otherwise if $\mathcal{F}(u) \leq 0$ we would have $x^2[u'(x)]^2 = 0$ for all $x \in (-1, 1)$ and by continuity of the derive we would obtain $u' \equiv 0$ so that $u \equiv \text{const}$ which is in contradiction to the boundary data $u(-1) = -1$ and $u(1) = 1$.

Especially this Weierstraß example toppled the calculus of variations into a Grundlagenkrise, and was later resurrected with new strength and vigor.

A minimizer of this problem is

$$\text{sgn}(x) := \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases},$$

so that we need to extend our comparison class. This will be performed in the lecture *Boundary and Eigenvalue Problems* held at KIT in the summer term 2024.

**Metapriniciple for the existence of a minimizer:**

**Theorem 8.1** (The direct method). Let $(\mathcal{X}, \| \cdot \|_{\mathcal{X}})$ be a normed space and $\mathcal{C} \subseteq \mathcal{X}$ be non-empty. Let $\mathcal{F} : \mathcal{C} \to \mathbb{R}$ be a functional which is

1. bounded from below and

2. coercive, that is

   for all $\{x_j\}_{j \in \mathbb{N}} \subseteq \mathcal{C}$ with $\lim_{j \to \infty} \|x_j\|_{\mathcal{X}} = \infty$ it holds $\lim_{j \to \infty} \|\mathcal{F}[x_j]\| = \infty$.

Moreover, suppose that on $\mathcal{X}$ there exists a convergence ‘$\sim \sim \sim \sim$’ such that
3. Every bounded sequence in $\mathcal{X}$ has a subsequence which converges with respect to $\sim \sim \sim \sim$, i.e. we have compactness,

4. $\mathcal{F}$ is lower semicontinuous with respect to $\sim \sim \sim \sim$, i.e., let $\{x_j\}_{j \in \mathbb{N}} \subset \mathcal{X}$ with $x_j \sim \sim \sim \sim x_0 \in \mathcal{X}$, then

$$\mathcal{F}[x_0] \leq \liminf_{j \to \infty} \mathcal{F}[x_j] = \lim_{j \to \infty} \left( \inf_{m \geq j} \mathcal{F}[x_m] \right),$$

5. $\mathcal{C}$ is closed with respect to $\sim \sim \sim \sim$.

Then there exists an $x \in \mathcal{C}$ that minimizes $\mathcal{F}$ over $\mathcal{C}$.

Proof. Since $\mathcal{F}$ is bounded from below we have

$$\inf_{\xi \in \mathcal{C}} \mathcal{F}[\xi] > -\infty.$$

By the definition of infimum we find a minimizing sequence $\{x_j\}_{j \in \mathbb{N}} \subseteq \mathcal{C}$ such that

$$\mathcal{F}[x_j] \xrightarrow{j \to \infty} d, \tag{8.1}$$

i.e. $\lim_{j \to \infty} \mathcal{F}[x_j] = d$. Since moreover $\mathcal{F}$ is coercive, the sequence $\{x_j\}_{j \in \mathbb{N}}$ is bounded and by 3. we find a convergent subsequence $\{x_{j_k}\}_{k \in \mathbb{N}}$ with respect to $\sim \sim \sim \sim$, and we denote by $x$ its limit, i.e.

$$x_{j_k} \sim \sim \sim \sim x \in \mathcal{X}.$$

By 5. $\mathcal{C}$ is closed with respect to $\sim \sim \sim \sim$, meaning that $x \in \mathcal{C}$. The theorem concludes by the lower semicontinuity of $\mathcal{F}$:

$$d = \inf_{\xi \in \mathcal{C}} \mathcal{F}[\xi] \leq \mathcal{F}[x] \leq \liminf_{k \to \infty} \mathcal{F}[x_{j_k}] \overset{(8.1)}{=} d.$$

All in all, $x$ is the desired minimizer.

---

in practise: $\mathcal{X}$ is a large function space $\sim \sim \sim \sim$ How regular is the minimizer?

Which type of convergence ‘$\sim \sim \sim \sim$’ do we have?

These questions will be addressed in higher courses like Boundary and Eigenvalue Problems or Calculus of Variations or Geometric Analysis etc.