RIEMANN-HILBERT PROBLEMS AND INTEGRABLE SYSTEMS
A PRELIMINARY VERSION

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INTRODUCTION

In recent years, the study of oscillatory Riemann–Hilbert problems has become very fruitful in the fields of integrable differential equations and integrable statistical models. The purpose of this set of lecture notes is to introduce the theory of Riemann–Hilbert problem method systemically and to prepare the reader with necessary working knowledge in the research frontier of this field.
CHAPTER 1. RIEMANN-HILBERT PROBLEMS AND DIRECT METHOD FOR INTEGRABLE SYSTEMS

1.0 Notation.

(1.0.1) In order to avoid the flooding of parentheses, when operator actions and multiplications take place in a row, we do them from right to left. For instance

\[ AbCd = A(bCd). \]

(1.0.2) Mathematicians are often in the dilemma of either abusing the expression \( \sin x^2 \) as the function defined by \( x \mapsto \sin x^2 \) or taking an obviously superfluous step to set \( f(x) = \sin x^2 \). A remedial measure often taken in mathematical literatures is to denote the function by \( \sin^2 \). However, in many complicated expressions the \( \cdot \)'s look misplaced and are found too tiny to be noticed, not speaking of the ugliness of their appearance. We will therefore sometimes use \( \diamond \) as the universal variable in place of \( \cdot \).

When this is still not satisfactory, such as in multi-variable case, we will also denote by \( \langle f(x) \rangle \) the function \( f \). In this case \( x \) is simply a dummy variable which should be viewed as a label to the variable rather than the variable itself. For example, the function \( f \) defined as \( f(x,t) = x^2 + e^{xt} \) may be denoted as \( \langle x^2 + e^{xt} \rangle \) while \( f(t) \) denoted as \( \langle x^2 \rangle + e^{\langle x^2 \rangle t} \) or \( \langle 0^2 \rangle + e^{\langle 0^2 \rangle t} \). This notation also makes it more convenient for defining functions and operators. For example, given a function \( v \) and an operator \( A \), \( A \diamond v \) would denote the operator defined by \( f \mapsto Afv \), while \( A \diamond v \) could be rather misleading.

(1.0.3) Two functions \( u \) and \( v \) are said to be continuous, analytic, etc. if it is continuous, analytic, etc. after a modification on a set of measure zero.

(1.0.4) \( L^p \), \( H^1 \) etc. are often used for matrix valued functions.

(1.0.5) The number \( 1 \) is also used for identity operators and \( I \) is used for identity matrices.

(1.0.6) If \( \mathcal{A} \) and \( \mathcal{B} \) are a Banach spaces. \( \mathcal{A} \diamond \mathcal{B} \) denotes the Banach algebra of bounded operators on \( \mathcal{A} \) and \( \mathcal{A} \rightarrow \mathcal{B} \) denotes the Banach space of bounded operator from \( \mathcal{A} \) to \( \mathcal{B} \).

(1.0.7) \( \mathcal{A}(D) \) denotes the space of (matrix) functions analytic on a set \( D \subset \mathbb{C} \). When \( D \) is open with a piecewise smooth boundary for each component, \( \mathcal{A}L^p(D) \) (\( \mathcal{A}L^p(D) \) denotes the space of functions in \( \mathcal{A}(D) \) with boundary values (from each component) in \( L^p \) \( = \frac{\mathcal{A}(D) \, dz}{(1 + |z|^2)} \)) sense.

1.1 Matrix Riemann-Hilbert problems—L^2 theory. The theory of complex variables relies in a quintessential way on the Cauchy integral representations of analytic functions. Let us first consider a bounded domain \( \Omega \subset \mathbb{C} \) whose boundary \( \partial \Omega \) is a rectifiable Jordan curve. We assume that the boundary is positively oriented in the sense that the domain \( \Omega \) lies on the left to its boundary.

Let us assume that \( f \) is analytic on \( \Omega \). A basic question is that under what additional conditions on \( f \) the Cauchy integral formula

\[
\begin{cases}
\frac{1}{2\pi i} \int_{\partial \Omega} \frac{f_+(\zeta)}{\zeta - z} \, d\zeta, & z \in \Omega \\
0 & z \in \Omega^-
\end{cases}
\]

holds, where \( f_+ \) denotes the boundary value of \( f \). A good place to find the results and references we state below is a book by Duren [D].
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We knew from the classical complex analysis that the Cauchy integral formula holds if \( f \) extends continuously to \( \hat{\Omega} \). However, in many applications, \( f \) does not extend continuously to \( \hat{\Omega} \) and we can only assume weaker conditions on \( f \).

Another case for the Cauchy integral formula to hold is when \( f \) is a Hardy \( H^1 \) function on the unit disk \( B^{\text{def}}\{ |z| = 1 \} \). Recall that a Hardy \( H^p \) function \( f \) on \( B \) is an analytic function on \( B \) such that

\[
\sup_{r<1} \|f\|_{L^p(\{|z|=r\})} < \infty.
\]

Every Hardy function \( f \) has nontangential boundary value \( f_+ \), a.e., which belongs to \( L^p(\partial B) \). Since \( f \) is uniquely determined by \( f_+ \), we sometimes also call \( f_+ \) the Hardy function. An analytic function \( f \) on \( B \) can be represented by its boundary value through Cauchy integral formula if \( f \in H^1 \).

These results of Hardy functions can be extended to our more general domain \( \Omega \) through Riemann mapping. A function \( f \) analytic in \( \Omega \) is said to be of class \( E^p \), \( 0 < p \leq \infty \) if there exists a sequence of rectifiable Jordan curves \( \Gamma_1, \Gamma_2, \ldots \) in \( \Omega \) tending to the boundary \( \partial \Omega \) in the sense that \( \Gamma_n \) eventually surrounds each compact subset of \( \Omega \), such that

\[
\lim_{n \to \infty} \|f\|_{L^p(\Gamma_n)} < \infty.
\]

It is also sufficient to consider only smooth curves \( \Gamma_n \) or even only the level curves of any Riemann mapping from \( \Omega \) to the unit disk. Therefore if \( f \in E^p \) and \( g \in E^q \), then \( f \circ g \in E^r \) when \( 1 \leq p, q, r \leq \infty \) and \( p^{-1} + q^{-1} = r^{-1} \). Again, \( f \in E^p(\Omega) \) has nontangential boundary value \( f_+ \), a.e., which belongs to \( L^p(\partial \Omega) \). The basic results which we need are (Chapter 10 [D]).

**Theorem 1.1.2.**

1. \( E^1(\Omega) = \{ \int_{\partial \Omega} \frac{\bar{g(z)}}{\zeta - \bar{z}} \, d\zeta = 0 \} \) for every \( z \in \mathbb{C} \setminus \hat{\Omega} \).

2. If \( g \in L^1(\Omega) \) and \( \int_{\partial \Omega} \frac{\bar{g(z)}}{\zeta - \bar{z}} \, d\zeta = 0 \) for every \( z \in \mathbb{C} \setminus \hat{\Omega} \), then

\[
f(\zeta) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{g(z)}{\zeta - z} \, d\zeta \in E^1(\Omega),
\]

and \( g = f_+ \).

The Jordan curve \( \partial \Omega \) can also be viewed as the negatively oriented boundary for a domain \( \Omega_- \) in the Riemann sphere which contains \( \infty \). \( E^p(\Omega_-) \) can be defined just as \( E^p(\Omega) \). Theorem above is true for \( E_0^p(\Omega_-) \) defined for \( f \in E^1 : f(\infty) = 0 \). (See Exercise).

The next question is what if \( \partial \Omega \) passes through \( \infty \) and is a rectifiable Jordan curve in the Riemann sphere with the spherical metric? The space \( E^p(\Omega) \) and \( E^p(\Omega_-) \) are still defined in the same way.

Since \( \infty \in \partial \Omega \), there is no difference in the results for \( E^p(\Omega) \) and \( E^p(\Omega_-) \). In the following we state only the results regarding \( E^p(\Omega) \). Again, every function \( f \in E^p(\Omega) \) has nontangential boundary value \( f_+ \in L^p(\partial \Omega) \). (See Exercise). The basic result here is.
Theorem 1.1.3. The Cauchy integral formula holds for every \( f \in E^p(\Omega) \), \( 1 \leq p < \infty \).

Proof. Without loss of generality, we assume that \( 0 \in \Omega_- \). Applying the conformal automorphism \( z \mapsto z^{-1} \), we see that the Cauchy integral formula is true for \( f \) on the domain \( \Omega \) iff it is true for \( z^{-1}f(z^{-1}) \) on the domain \( \Omega^{-1} \{ z^{-1} : z \in \Omega \} \), i.e.

\[
(1.1.3a) \quad z^{-1}f(z^{-1}) = \frac{1}{2\pi i} \int_{\partial \Omega^{-1}} \frac{\zeta^{-1}f_\Omega(\zeta^{-1})}{\zeta - z} \, d\zeta, \quad z \in \partial \Omega.
\]

For any rectifiable curve \( \Gamma \) in \( \Omega \), we have by Hölder inequality

\[
\frac{1}{\nu_{n-1}^\Gamma} \int_{\Gamma} |z^{-1}f(z^{-1})| |dz| = \int_{\Gamma} |z^{-1}f(z)| |dz| \leq c\|f\|_{L^p(\Gamma)}. \quad 1 \leq p < \infty.
\]

Thus if \( f \in E^p(\Omega) \), \( 1 \leq p < \infty \), then \( \hat{f}(\hat{\gamma}^{-1}) \in E^1(\hat{\gamma}^{-1}) \). This gives the Cauchy integral formula (3).

The \( E^p \) space, however, is not invariant under the Mobius transforms \( z \mapsto \frac{az + b}{cz + d} \), \( ad - cb \neq 0 \), on \( \hat{\mathbb{C}} \) (see (Q1.2)). The metric for the Riemann sphere is \( 2|dz|/(1 + |z|^2) \), induced by the stereographic projection from the unit sphere \( S^2 \) to \( \hat{\mathbb{C}} \cup \{ \infty \} \) (see Exercise (Q1.1)).

We denote \( L^p(\Gamma) = L^p(\Gamma, 2|dz|/(1 + |z|^2)) \). The \( L^p \) space is invariant under Mobius transforms.

Since \( \hat{\mathbb{C}} \) is compact, \( L^p_{\text{loc}}(\Gamma) = L^p(\Gamma) \) and \( L^p \subset L^q \), for \( p > q \). However, if \( \infty \not\in \Gamma \), \( L^p(\Gamma) = L^p(\Gamma) \) with equivalent norms. We now consider a larger space \( E^p \).
defined in the same way as $E^p$ except we replace the $L^p$ norm in (1.1.1a) by the $L^p$. Then clearly $E^p$ is invariant under the Mobius transform and the compactness of the sphere implies that $E^p \subset E^q$ if $p > q$. Of course, $E^p = E^p$ when $\infty \not\in \partial \Omega$.

Two examples of functions in $E^p(\mathbb{C}^+)$ but not in $E^p(\mathbb{C}^+)$ are $e^{iz}$ and $z^{-1/4}$, where $\mathbb{C}^+$ denotes the upper Riemann sphere.

When $\infty \not\in \Gamma$, we cannot use the standard Cauchy formula for functions in $E^p$. Instead, we have the following generalized Cauchy formula for $f \in E^p$, $1 \leq p \leq \infty$.

\begin{equation}
    f(z) = \frac{1}{2\pi i} \int_{\partial \Omega} f_+(\zeta) \frac{(z - a)d\zeta}{(\zeta - a)(\zeta - z)}, \quad z \in \Omega,
\end{equation}

where $a$ is any point in $\Omega$. If we simply use the transform $z \mapsto (z - a)^{-1}$ and the fact that $E^p$ is invariant under this transform, we reduce this Cauchy formula to (1.1.3a) for $\Omega^{-1}$ which is bounded in $\mathbb{C}$. The differential form $\frac{(z-a)d\zeta}{2\pi i(z-a)(\zeta - z)}$ is in fact a generalized Cauchy kernel which has two simple poles at $\zeta$ and $a$ while the standard Cauchy kernel has two simple poles at $\zeta$ and $\infty$.

Let the contour $\Gamma$ (see Figure 1.1a), as a closed subset, be the union of a finite number of oriented smooth curve segments on the Riemann sphere $\hat{\mathbb{C}}$ such that $\hat{\mathbb{C}} \setminus \Gamma$ has only a finite number of connected components. and define $L^p(\Gamma) = L^p(\Gamma, 2|dz|/(1 + |z|^2))$, $1 \leq p \leq \infty$.

For the study of Riemann–Hilbert problems, the set $\Omega$ for analytic functions is $\mathbb{C} \setminus \Gamma$ for a contour $\Gamma$. In general, a connected component of $\Omega$ need not be a Jordan domain. However, we can always augment the contour $\Gamma$ to a contour $\Gamma'$ by adding some oriented smooth curves to $\Gamma$ so that each connected component of $\mathbb{C} \setminus \Gamma'$ is a Jordan domain. We say that an analytic function $f$ is in the space $E^p(\Omega)$ if $f$ is uniform in a neighborhood of $\infty$ and $f \in L^p(\partial \Omega)$ based on the $L^p$ theory of Cauchy integral operators (see below). Thus if $f \in E^p$ and $g \in E^q$, then $fg \in E^p$ for $p^{-1} + q^{-1} = r^{-1}$ and $1 < p, q \leq \infty$. Similar results hold for $E^p$.

Some recent studies of integrable systems involve solutions of RH problems which oscillate or even grow at the rate $O(z^{1/4})$ as $|z| \to \infty$. Hence the standard $L^2$ theory of RH problems is not sufficient. It is convenient in practice to define the RH problem on the Riemann sphere in a way that it is invariant under Mobius transforms. In this section, we introduce the $L^2$ theory of RH problems on the Riemann sphere. The $L^p$ theory will be introduced in the next chapter, together with the Gohberg–Krein factorization theory.

We call an $n \times n$ matrix valued function $v$ on $\Gamma$ a jump matrix (or condition) for a Riemann–Hilbert (factorization) problem if $v \in L^\infty(\Gamma)$. We will always assume Condition (C1.1.3c) for $v, v^{(k)}$, $v_{\pm}$, etc. unless otherwise stated.

The $L^2$ theory of the RH problem for $(v, \Gamma)$ is defined as follows.

We call a $k \times n$ matrix function $m \in E^2(\hat{\mathbb{C}} \setminus \Gamma)$ a solution of the RH problem $(v, \Gamma)$ or sometimes denoted by $(m, v, \Gamma)$, if

\begin{equation}
    m_+ = m_- v, \quad \text{on } \Gamma.
\end{equation}
If furthermore, $m$ is $n \times n$ and $m^{-1} \in \mathbb{E}^2$, then $m$ is said to be a fundamental solution of the RH problem and when it exists the RH problem is said to be solvable.

This definition of RH problems is clearly invariant under Mobius transforms on $\hat{\mathbb{C}}$.

A RH problem need not have a fundamental solution (see exercise Q). Nor is a fundamental solution unique unless the value of $m(z)$ at a particular point $z = a$ is pre-fixed.

An $n \times n$ matrix solution $m$ is said to be normalized at $a \in \hat{\mathbb{C}}$ along a curve $\gamma$ if $m(z) \rightarrow I$, as $z \rightarrow a$ along $\gamma$, where $I$ denotes the identity matrix.

**Proposition 1.1.3e.** Let $m$ be a fundamental solution of the RH problem for $(\nu, \Gamma)$ and $m_1$ be another solution of the same RH problem. Then $m_1 = H m$ for some constant matrix $H$.

If furthermore $m$ is normalized at $a$, then $m_1(a)$ exists and equals $H$.

**Proof.** Consider $H(z) = m_1(z)(m(z))^{-1}$. Then $H \in \mathbb{E}^1$ with boundary values $H_+ = H_-$. Thus $H$ is analytic on $\hat{\mathbb{C}}$ and therefore a constant. The rest of the proposition follows trivially. □

As a consequence a fundamental solution normalized at a given point is unique.

The following very useful proposition says sometimes a normalized solution is automatically fundamental. It applies particularly well in the study of integrable systems where we often have $2 \times 2$ case and $\det m = 1$.

**Proposition 1.1.4.** Let $m$ be a solution with $\det m(a) \neq 0$ for some $a \notin \Gamma$ and assume that $m \in \mathbb{E}^2$, $p \geq \max(2, n)$. If $\left(\det \nu^{-1}\right)$ admits a fundamental solution $m_1 \in \mathbb{E}^2$, $\frac{1}{q} \leq \frac{1}{2} - \frac{n-1}{p}$, then $m_1$ is a fundamental solution.

**Proof.** Since $\det m \in \mathbb{E}^{2+n}$ by Hölder's inequality and $\frac{1}{q} \leq \frac{1}{2} - \frac{n-1}{p}$, $\det m m_1 \in \mathbb{E}^2$. Then it follows from the fact that $\det m_1 = \det \nu^{-1}$ and $\det m m_1$ must be analytic on $\hat{\mathbb{C}}$ and hence a constant. Thus $\left(\det m^{-1}\right) = \left(\det m(a)\right)^{-1} m_1^{-1}(a) m_1$. Hence $m^{-1} \in \mathbb{E}^{1 - \frac{n-1}{p}} \subset \mathbb{E}^2$. □

**Remark 1.1.5.** In the $2 \times 2$ case with $\det \nu = 1$, $q = \infty$ and hence we may choose $p = 2$ which is not an extra condition. For RH problems for decomposing algebras which we study next, the only condition we need to check for Proposition 1.1.4 is $\kappa(\nu) = 0$. In fact, by the results of next chapter, when $\nu$ is continuous (this does not imply that $m$ is continuous (Q1.)), $m$ and $\det m \in \mathbb{E}^2$ if and only if they belong to $\mathbb{E}^p$ for all $1 < p < \infty$. Hence $m^{-1} \in \mathbb{E}^2$ also.

The most important operators related to RH problems are Cauchy (integral) operators.

We define $C_\Gamma$ for functions $f \in L^p$, $1 < p < \infty$ as

$$
(C_\Gamma f)(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \notin \Gamma.
$$

$C_\Gamma f$ is clearly analytic on $\hat{\mathbb{C}} \setminus \Gamma$. Let $\Gamma'$ be also a contour. The following are some basic results of harmonic analysis regarding Cauchy integral operators.

(1.1.7) $C_\Gamma$ is bounded from $L^p$ to $L^p(\hat{\mathbb{C}} \setminus \Gamma)$.
(1.1.8) The operator $C_{\Gamma}^{\pm}$ defined by $f \rightarrow$ the boundary values $(C_{\Gamma} f)_{\pm}$ on $\Gamma$ is bounded from $L^p(\Gamma)$ to $L^p(\Gamma')$.

(1.1.9) $C^+ - C^- = 1$. $C^{\pm} \overset{\text{def}}{=} C^{\pm}_{\Gamma} \overset{\text{def}}{=} C^{\pm}_{\Gamma} \overset{\text{def}}{=} C^{\pm}.$

Figure 1.1b

The operators $C_{\Gamma}^{\pm}$ are also called Cauchy (integral) operators. To explore more properties of $C^{\pm}$, we call a contour $\Gamma$ a complete contour if

(C1.1.10a) $\Omega_{\Gamma} \overset{\text{def}}{=} \mathcal{C} \setminus \Gamma$ is the disjoint union of two, possibly disconnected, open regions $\Omega_+$ and $\Omega_-.$

(C1.1.10b) $\Gamma$ may either be viewed as the positively oriented boundary for $\Omega_+$ or as the negatively oriented boundary for $\Omega_-.$

(C1.1.10c) For each component $\omega$ of $\Omega$, $\partial \omega$ has no self-intersections.

We remark that condition (C1.1.10) is not essential, but it provides some convenience for further study. Besides, any contour $\Gamma$ can be augmented to a complete contour (see (Q1.)).

A self-intersection
In this figure, $\partial \omega$ has a self-intersection. A augmented contour is formed by adding a small (dashed) circle around the self-intersection so that condition (C1.1.10c) is satisfied. Now we have that if $\Gamma$ is a complete contour, then

\[(1.1.11) \quad C^+ \text{ and } C^- \text{ are complementary projections in the sense that } (C^+)^2 = C^+, \quad (C^-)^2 = C^-, \quad \text{and } C^+ - C^- = 1.\quad \text{(Q1.)}
\]

$C_\Gamma f$ may be viewed as a solution of the "additive" RH problem for $(f, \Gamma)$: $f_+ - f_- = C^+ f - C^- f = f$. Exercise (Q.1.) shows that this additive RH problem is equivalent to a "multiplicative" RH problem in a triangular form. On the other hand, a scalar RH problem can be converted into an additive RH problem by the logarithm transform.

The Cauchy operators $C_\Gamma^\pm$ are not bounded on $L^p$ when $\infty \in \Gamma$. However, in this case, the generalized Cauchy operator $C_{\Gamma, \alpha}$ defined as $C_{\Gamma, \alpha} f = (z - \alpha)C_\Gamma (z - \alpha)^{-1} f$ for some $\alpha \not\in \Gamma$ is bounded from $L^p(\Gamma)$ to $E^p(\Omega)$ for $1 < p < \infty$. The operators $\pm C_\Gamma^\pm$ are bounded in $L^p(\Gamma)$ for $1 < p < \infty$ and are complementary projections when $\Gamma$ is complete.

We call a solution $m$ of the RH problem $(\nu, \Gamma)$ normalized at $\infty$ in $L^2$ sense if $m - I \in \text{ran} \, C_{\Gamma}$. This type of solutions are very common in the study of integrable systems.

Finding solutions normalized at $\infty$ in $L^2$ sense is equivalent to solving a singular integral equation by the following theorem.

**Theorem 1.1.12.** Assume $\nu - I \in L^2$ (in addition to (C1.1...)). If $\mu$ is a solution of the following equation

\[(1.1.13) \quad \mu - h = C_\nu \mu = C^- \mu (\nu - I) \quad \text{for some } h \in L^2(\Gamma) + L^\infty(\Gamma),
\]

for some $h \in L^2(\Gamma) + L^\infty(\Gamma)$, then

\[(1.1.14) \quad H \overset{\text{def}}{=} C_\Gamma \mu (\nu - I)
\]
satisfies

(C1.1.15a) $H \in E^2(\mathcal{C} \setminus \Gamma)$, (C1.1.15b) $H_+ + h = (H_+ - h)\nu$ and $H_- + h = \mu$.

(C1.1.15c) $H \to 0$ as $z \to \infty$ along all curves nontangential to $\Gamma$.

Conversely, if a function $H$ satisfies (C1.1.15a) and (C1.1.15b), and if $H$ converges to $0$ as $z \to \infty$ along any curve nontangential to $\Gamma$, then $\mu \overset{\text{def}}{=} H_- + h$ is a solution of equation (1.1.13).

**Proof.** Assume that $\mu$ is a solution of (C1.1.13). Clearly, $H$ satisfies (C1.1.15a) and (1.1.15c). Also $H_- = \mu - h$ by the definition of $C_\nu$ and $H_+ + h = h + C^+ \mu (\nu - I) = h + \mu (\nu - I) + C^- \mu (\nu - I) = (H_+ + h)(\nu - I) = H_+ \nu - H_-$ and therefore $H - H_1$ must be a constant. Hence $H - H_1 \to 0$ along a curve, hence $H = H_1$ and $\mu \overset{\text{def}}{=} H_- + h$ is a solution of equation (1.1.13). □

Clearly, when $h = I$, the matrix function $m \overset{\text{def}}{=} H + I$ is a solution of the RH problem $(\nu, \Gamma)$ normalized at the $\infty$ in $L^2$ sense. When $h = 0$, $m \overset{\text{def}}{=} H$ is called the
vanishing solution of the RH problem. A vanishing solution represents an element in \( \ker(1 - C_+). \)

Another interesting question is that when \( C^\pm f \) have continuous boundary values, (Q1.1) gives an example that even for a continuous function \( f \) on a smooth complete contour, \( Cf \) need not have continuous boundary values. We need certain smoothness for \( f \).

Assume that \( \Gamma \) is a complete contour without self-intersection. Since the Cauchy operators commute with differentiation, thus \( (Cf)' = Cf' \) and \( (C^\pm f)' = C^\pm f' \) provided \( f' \in L^2(\Gamma) \) in distribution sense (Q1.). We define the Sobolev space \( H^1(\Gamma) \) by the inner product

\[
(f, g)_{H^1} = \int_{\Gamma} \text{tr}(fg^* + f'(g')^*) |dz|
\]

for scalar or matrix functions \( f \) and \( g \). Then \( C_f \) has \( H^1 \) boundary values and \( \pm C^\pm \) are bounded complementary projections in \( H^1 \). It is easy to show that \( H^1 \) is contained in the Hölder space \( H^{1/2}(\Gamma) \) (Q1.).

The Sobolev space \( H^1 \) is a Banach algebra in the sense that the product map \( <f, g> \rightarrow fg \) is continuous from \( H^1 \times H^1 \rightarrow H^1 \) (Q1.). This important property makes it very convenient to work with \( H^1 \) space in the study of RH problems. A Banach algebra of continuous functions with the Cauchy operators bounded in it is called a decomposing algebra. When \( \alpha \in \Gamma \), the algebra \( H^1 \) does not have an identity element. In fact, none of the elements in \( H^1 \) is invertible. We therefore extend it to the Hilbert space direct sum \( H^1 \oplus M(\mathbb{C}) \), where \( M(\mathbb{C}) \) denotes the algebra of all constant \( \mathbb{C} \times \mathbb{C} \) valued functions with standard matrix inner product. The Cauchy integral operators \( C^\pm \) are extended to be bounded operators in \( H^1 \) by setting \( C^+ f = f \) and \( C^- f = 0 \) for \( f \in M(\mathbb{C}) \). A Banach algebra is called inverse closed if furthermore \( f^{-1} \) is in the algebra whenever \( f \) is and \( f^{-1} \) is nonsingular for \( f \in \Gamma \). \( H^1 \) is known to be an inverse closed decomposing algebra. The Hölder spaces are other examples of inverse closed decomposing algebras. However, working with a Hölder space is not as convenient because it is not separable (ref).

We also define a Banach algebra \( H^1(\Gamma) \) with its inner product in the same form of (1.1.16) but with \( dz \) in \( f, g' \) and \( |dz| \) replaced by \( ds \); furthermore \( f^{-1} \) is in the algebra whenever \( f \) is and \( f^{-1} \) is nonsingular for \( f \in \Gamma \). \( H^1(\Gamma) \) is known to be an inverse closed decomposing algebra. The Hölder space \( H^0(\Gamma) \) is defined with the metric \( ds \).

With generalized Cauchy operators, Theorem 1.1.12 can be extended to

**Theorem 1.1.19.** Let \( \alpha \notin \Gamma \). If \( \mu \) is a solution of the following equation

\[
(1.1.20) \quad \mu - h = C_\alpha \mu \quad \text{def} \quad C_\alpha \mu(v - I)
\]

for some \( h \in L^2(\Gamma) \), then

\[
(1.1.21) \quad H^\prime \mu = C_\alpha \mu(v - I)
\]

satisfies

(C1.1.22a) \( H \in E^2 \),
(C1.1.22b) \( H_\alpha + h = (H_\alpha + h) (v - I) \) and \( H_\alpha + h = \mu \),
(C1.1.22c) \( H(\alpha) = 0 \).
Conversely, if a function $H$ satisfies (C1.1.22abc), then $\mu \overset{\text{def}}{=} H^- + h$ is a solution of Equation (1.1.20).

Assume that $\Gamma$ is smooth and $\nu(z)$ is a continuous scalar function on $\Gamma$ which vanishes nowhere. Denote by

\begin{equation}
(1.1.23) \quad W_T(\nu) = \int_\Gamma d \arg \nu(z)
\end{equation}

the winding number of $\nu$ along $\Gamma$.

**Theorem 1.1.24.** Assume that $\Gamma$ is a complete contour without self-intersections and $\nu \in H^1$ (for the Hölder space $H^\alpha$) is a non-vanishing scalar function. Then the RH problem $(\nu, \Gamma)$ has a fundamental solution $m \in H^1$ ($H^\alpha$) if and only if $W_T(\nu) = 0$. Furthermore, when this is true,

\begin{equation}
(1.1.25) \quad m = e^{C_{\Gamma, \nu} \log \nu}
\end{equation}

is the solution normalized at $a \not\in \Gamma$.

**Proof.** The “only if” part following directly from the facts that $W_T(m_+) = 0$ and $W_T(m_-) = W_T(m) + W_T(\nu)$.

For the “if” part, note that if $\kappa(\nu) = 0$, a log $\nu$ can be defined as a single valued function in $H^1$ ($H^\alpha$) (Q1.1). One checks easily that $m = e^{C_{\Gamma, \nu} \log \nu}$ and $m^{-1}$ are both in $H^1$ ($H^\alpha$) and satisfies the jump condition.

**Theorem 1.1.26.** Assume that $\Gamma$ is a complete contour without self-intersections and $\nu$ a continuous scalar function with $W_T(\nu) = 0$. Then for any $a \not\in \Gamma$, the RH problem $(\nu, \Gamma)$ has a fundamental solution $m$ normalized at a such that $m, m^{-1} \in L^p(\Gamma)$ for all $1 < p < \infty$. In addition, the solution $m$ can again be rewritten as (1.1.25).

**Proof.** Fix $1 < p < \infty$. Since any continuous function can be approximated by $H^1$ functions in $L^\infty$ norm, we can write $\nu = \nu^{(1)} + \nu^{(2)}$ such that

\begin{equation}
(1.1.27a) \quad \nu^{(1)} \in H^1, \quad \nu^{(1)}(z) \text{ is nonsingular for all } z \in \Gamma, \quad \kappa(\nu^{(1)}) = 0.
\end{equation}

\begin{equation}
(1.1.27b) \quad \|\nu^{(2)} - 1\|_{L^\infty} < 1/\|C_{\Gamma, \nu}^{-1}\|_{L^p}.
\end{equation}

Clearly, the equation

\[ \mu = I + C_{\Gamma, \nu}^{-1} \kappa(\nu^{(1)}) - 1 \]

has a unique $L^p$ solution by condition (1.1.27b). Thus $m^{(2)} \overset{\text{def}}{=} C_{\Gamma, \nu} m^{(1)} - 1$ has $L^p$ boundary values which satisfy the jump condition $\nu^{-1}$ and is normalized at $a$. On the other hand, Theorem 1.1.24 and (1.1.27a) guarantees an $H^1$ fundamental solution $m^{(1)}$ for the RH problem $(\nu^{(1)}, \Gamma)$. Thus $m = m^{(1)} m^{(2)}$ satisfies the jump condition $\nu$.

Similarly, we can find $\tilde{m}$ normalized at $a$ with $L^p$, $1/p' + 1/p = 1$, boundary values satisfying the jump condition $\nu^{-1}$. Thus $m \tilde{m}$ is analytic on $\mathbb{C}$ and hence a constant. This constant must be 1 because of the normalization condition of $m$ and $\tilde{m}$ at $a$. This says $m$ has an $L^p$ inverse. An $L^p$ solution with $L^p$ inverse must be unique. Finally, since $L^p \subset L^q$ for $p \geq q$, all these $L^p$ solutions must be the same for all $1 < p < \infty$. To show that $m$ can be written as (1.1.25), see (Q2.1).

**Remark 1.1.28.** In inverse scattering theory, $\Gamma$ often has self-intersections and the continuity (smoothness) of $\nu$ is not in the conventional sense. We will give the definition (smoothness) in Chapter 2.
1.2 Lax Pairs and Integrable Evolutions. In this section we derive Lax pairs from fundamental solutions of certain RH problems.

Let $J(z)$ be a rational matrix function. Consider the RH problem for

$$
(\mathcal{e}^{x \mathcal{ad} J(z)} v(z), \Gamma)
$$

with a parameter $x$, where $\mathcal{ad} J$ stands for the operator $[J, \cdot] \overset{\text{def}}{=} DJ - JD$ on matrices and $e^{x \mathcal{ad} J} v = e^{xJ}v e^{-xJ} \overset{\text{def}}{=} e^{Jx}v$. Assume that this RH problem has a fundamental solution $m$ with $m_z = \partial m/\partial z \in E^2(\mathbb{C} \setminus \Gamma)$. Set $\psi = m e^{xJ}$ and consider the function

$$
U = U(x,z) \overset{\text{def}}{=} \psi_z \psi^{-1} = m_z m^{-1} + m J m^{-1}
$$

Clearly $U_+ = U_-$ on $\Gamma$ and $U$ is rational in $z$ and has the same poles (with the same orders) as $J$ does.

For a meromorphic function $f$, denote by ord$_z f$ the order of $f$ at $z' \in \mathbb{C}$, i.e.

$$
\text{ord}_z f = n \text{ if for some } c \neq 0 \text{ as } z \to z'.
$$

$$
f(z) \sim \begin{cases} 
  c(z-z')^n, & z' \in \mathbb{C} \\
  c z^{-n}, & z' = \infty.
\end{cases}
$$

Let us also denote by $P_J$ the set of poles of $J$. Clearly, for $z' \in J_p$, $\text{ord}_z (U - J)$ satisfies

(C1.2.2a) \quad $\text{ord}_z (U - J) \geq \text{ord}_z J$.

and

(C1.2.2b) \quad If $m(x,z')$ commutes with $J(z)$, then $\text{ord}_z (U - J) > \text{ord}_z J$.

We now consider the RH problem

$$
(\mathcal{e}^{x \mathcal{ad} \tilde{J}(z)} \tilde{V}(z), \Gamma)
$$

with two parameters $x$ and $t$, where $\tilde{J}(z)$ commutes with $J$. Again assume that there be a fundamental solution $\tilde{m} = m(x,t,z)$ normalized at $\infty$. Also assume that $m_x$ and $m_t$ both have $L^2$ boundary values. We will then derive a pair of equations

(1.2.4a) \quad $\psi_x = U(x,t,z) \psi$

(1.2.4b) \quad $\psi_t = V(x,t,z) \psi$.

where both $U$ and $V$ are rational in $z$ and $U$ satisfies (C1.2.2a) while $V$ satisfies a similar condition with $\tilde{J}$ in place of $J$ in (C1.2.2a). Equations (1.2.4a) and (1.2.4b) are the so-called Lax pair. Assume further that $m$ has continuous second order derivatives in $x$ and $t$, then compatibility conditions of these two equations yields

$$
U_t - V_x = [V, U] \equiv VU - UV.
$$

Since both sides of this equation are rational in $z$, comparing their coefficients, we obtain a system of partial differential equations, which are called an integrable system. To determine whether a given equation is integrable one needs to rewrite the equation in the form of (1.2.3). This is in general a very difficult problem
It usually works other way around: one starts from the Lax pair to see what
\textit{equations} it generates using the asymptotic expansions of $\psi$ or $m$ in $z$ near the
poles of $J$ and $\bar{J}$. See Example 1.2.10 below for more explanations.

In a more general setting, one considers RH problems
\begin{equation}
(\mathbf{e}^{\sum_{k=1}^{N} x_k \text{ad} H_k(z)} v(z), \Gamma).
\end{equation}
where $J_k(z)$, $k = 1, \ldots, N$ are mutually commuting rational functions. We derive
in analogy to (1.2.4ab)
\begin{equation}
\partial_{x_k} \psi_k = U_k(x_1, \ldots, x_N; z), \quad k = 1, \ldots, N.
\end{equation}
The related integrable system may be either written as
\begin{equation}
(U_j)_{x_k} - (U_k)_{x_j} = [U_k, U_j], \quad j, k = 1, \ldots, N.
\end{equation}
or in the so-called zero curvature form
\begin{equation}
[\partial_{x_k} - U_k, \partial_{x_j} - U_j] = 0.
\end{equation}
The matrices $J$ and $\bar{J}$ can also be allowed to depend on $x$ and $t$ respectively.
The RH problem (1.2.3) should be replaced by
\begin{equation}
(\mathbf{e}^{\text{ad} f', J} - \text{ad} f' \psi, \Gamma).
\end{equation}

The following example shows how one can \textbf{nonlinearize} a linear equation through
a RH problem to obtain a nonlinear integrable system. In this respect, the method
of solving an integrable system may be viewed as a \textbf{nonlinear} superposition op-
posed to a linear superposition.

\textbf{Example 1.2.10.} Nonlinear Schrödinger equation, modified KdV equation, and hi-
erarchies.

The linear equation
\begin{equation}
iQ_t + Q_{xx} = 0
\end{equation}
has a plane wave solution of the form
\begin{equation}
et{z^2 - itz^2},
\end{equation}
with a parameter $z$. A general decay in $x$ solution can be obtained through a linear
superposition, known as the Fourier transform. Now let us consider the RH problem
\begin{equation}
(m, e^{ix^2 - itz^2} \text{ad} \sigma \psi, \Gamma), \quad \sigma = \begin{pmatrix}
\frac{i}{2} & 0 \\
0 & -\frac{i}{2}
\end{pmatrix}
\end{equation}

\text{normalised at $\infty$ and set}
\begin{equation}
\psi = me^{ix^2 - itz^2} \sigma.
\end{equation}

Thus we have the Lax pair based on (C1.2.2b)
\begin{equation}
\partial_x = 12\sigma \psi + g\psi.
\end{equation}
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(1.2.11b) \[ \psi_t = -iz^2 \sigma \psi + (q_0 - zq_1) \psi. \]

by (1.2.2). This Lax pair can also be written in the form

(1.2.11c) \[ m_z = [i \sigma, m] + qm. \]

(1.2.11d) \[ m_t = [-i z^2 \sigma, m] + (q_0 - zq_1)m. \]

Substituting in (1.2.11cd) the asymptotic expansion

(1.2.11e) \[ m = I + m_1/z + m_2/z^2 + m_3/z^3 + \ldots \quad \text{as } z \to \infty. \]

we obtain

(1.2.12a) \[ 0 = [i \sigma, m_1] + q. \]

(1.2.12b) \[ m_{1z} = [i \sigma, m_2] + qm_1. \]

(1.2.12c) \[ 0 = [-i \sigma, m_1] - q_1. \]

(1.2.12d) \[ 0 = [-i \sigma, m_2] + q_0 - q_1 m_1. \]

For a square matrix \( A \), we denote by \( A_{\text{diag}} \) and \( A_{\text{off}} \) the diagonal part of \( A \) and the off-diagonal part of \( A \) respectively. By (1.2.12ac).

(1.2.12e) \[ q = q_1 = \begin{pmatrix} 0 \\ -\tilde{Q} \end{pmatrix}, \quad Q = -i(m_1)_{12}, \quad \tilde{Q} = -i(m_1)_{21}. \]

By (1.2.12d) and (1.2.12e).

(1.2.12f) \[ (q_0)_{\text{diag}} = \begin{pmatrix} 0 & Q \\ -\tilde{Q} & 0 \end{pmatrix}, \quad (m_{1z})_{\text{diag}} = \begin{pmatrix} (m_1)_{12} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & iQ \\ i\tilde{Q} & 0 \end{pmatrix} = 2i \tilde{Q} \sigma. \]

By (1.2.12bde).

(1.2.12g) \[ m_{1z} = q_0. \]

and hence

(1.2.12h) \[ (q_0)_{\text{off}} = (m_{1z})_{\text{off}} = \begin{pmatrix} 0 & iQ_z \\ i\tilde{Q}_z & 0 \end{pmatrix}. \]

Now combining (1.2.12f) and (1.2.12h), we have

(1.2.12i) \[ q_0 = \begin{pmatrix} i\tilde{Q}Q & iQ_z \\ i\tilde{Q}_z & -i\tilde{Q} \end{pmatrix}. \]

Finally, one verifies directly that (1.2.5) gives the following nonlinear Schrödinger equation

(1.2.13) \[ \begin{cases} iQ_t + Q_{xx} + 2Q^2\tilde{Q} = 0 \\ i\tilde{Q}_t - \tilde{Q}_{xx} - 2Q\tilde{Q}^2 = 0 \end{cases}. \]
Two interesting cases in physics are

1. Focusing nonlinear Schrödinger equation ($\dot{Q} = Q$).

\[(1.2.13a) \quad i Q_t + Q_{xx} + 2|Q|^2 Q = 0.\]

2. Defocusing nonlinear Schrödinger equation ($\dot{Q} = -Q$).

\[(1.2.13b) \quad i Q_t + Q_{xx} - 2|Q|^2 Q = 0.\]

When the jump matrix has sufficient decay to $I$, from (1.3.13) with $h = I$ and (1.2.12a), we have

\[m_1 = \lim_{z \to \infty} z(m - I) = -\frac{1}{2\pi i} \int_{\Gamma} m_-(x, t, z)e^{ixz - itz^2} \text{ad}_\sigma(v(z) - I)dz.\]

Thus

\[q = \begin{pmatrix} 0 & Q \\ -\dot{Q} & 0 \end{pmatrix} = -\frac{1}{2\pi i} \int_{\Gamma} m_-(x, t, z)e^{ixz - itz^2} \text{ad}_\sigma(v(z) - I)dz.\]

This may be viewed as a nonlinear superposition formula. If $m_-$ is replaced by $I$, this formula reduces to a linear superposition. We point out here that in linear case, all solutions decay in $x$ can be obtained through such a superposition on $\mathbb{R}$, but this is not the case for nonlinear integrable equations. One of the main motivations to the study of integrable systems is that they possess the so-called soliton solutions (see section 1.4) which are obtained through nonlinear superpositions on some circles away from the real axis $\mathbb{R}$.

In order that the asymptotic expansion (1.2.11e) exists, the RH problem must be solvable and the jump matrix must have sufficient decay (to $I$) at $z = \infty$. However, the derivation of the Lax pairs and corresponding integrable equations is completely an algebraic procedure irrelevant to the analytic behaviours of the RH problem. The result of the derivation is also purely algebraic.

The set of all integrable equations which share the same first equation of the Lax pairs, under the condition

(C1.2.) Every pole of $\text{ad} \dot{J}$ is also a pole of $\text{ad} J$, is called a hierarchy, usually named by the most famous equation in the hierarchy. For instance, any integrable equation with (1.2.11a) as the first equation of its Lax pair may be called an equation in the nonlinear Schrödinger equation (NLS) hierarchy.

When (C1.2.) is not satisfied, we call the set of equations a singular hierarchy.

In the following, we show how to derive the hierarchy systematically through the example of NLS hierarchy.

Consider the RH problem

\[(m, e^{ixz \text{ad}_\sigma + tJ(z)}v, \Gamma),\]

normalized at $\infty$, where $J(z)$ is a polynomial and diagonal. We have the corresponding Lax pair.

\[(1.2.14a) \quad m_x = [iz\sigma, m] + q(x, t)m,\]
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\( (1.2.14b) \quad m_t = [\tilde{J}(z), m] + \tilde{q}(x, t, z)m. \)  

Based on (C1.2.2b), \( \tilde{q}(z) \) is a polynomial of degree 1 less than that of \( \tilde{J}(z) \). We rewrite (1.2.14b) as

\( (1.2.14c) \quad m_t m^{-1} = \tilde{J} - m \tilde{J} m^{-1} + \tilde{q}. \)

Since the LHS is of order \( O(\frac{1}{z}) \) as \( z \to \infty \), \( \tilde{q} \) and the asymptotic expansion of \( m_t m^{-1} \) at \( z = \infty \) are completely determined by the asymptotic expansion of \( G \). We define

\( G \triangleq m \tilde{J} m^{-1} = G_0 + G_1/z + G_2/z^2 + \cdots. \)

In particular, the evolution equation can be obtained as

\( q_t = i[\sigma, m_{1t}] = -i[\sigma, G_1]. \)

We notice that \( G \) satisfies the following equation

\( G_z = iz[\sigma, G] + [q, G]. \)

Since this equation is linear in \( G \), we may assume first that \( \tilde{J} \) is a constant diagonal matrix \( A \). Since adding to \( A \) by the diagonal matrix of the form \( cI \) makes no change on (1.2.14b), we may assume \( A = i\sigma \). By the linearity of (4.14), we may as well assume first that \( A = i\sigma \). Substitute the asymptotic expansion (5.) into (1.), we obtain

\( 0 = i[\sigma, G_0]. \)

\( G_{j+1} = i[\sigma, G_j] + [q, G_j]. \)

Now we can determine recursively, in the order, \( G_0 = i\sigma \), \( G_{j, \text{off}} \), \( G_{j, \text{diag}} \) and then \( G_{j+1, \text{diag}} \) through integration. By (5.19),

\( G_{j+1, \text{off}} = \text{ad} (z + i \text{ad} q \text{P}_{\text{diag}}) G_j. \)

where \( \text{P}_{\text{diag}} \) denotes the projection operator to the diagonal parts of matrices. Apply the same again with \( j + 1 \) in place of \( j \), we see that

\( G_{j+1, \text{diag}} = \partial_z^{-1} \text{ad} q G_{j+1, \text{off}}. \)

Hence

\( -i \text{ad} \sigma G_{j+1} = (-\partial_z + \text{ad} q \partial_z^{-1} \text{ad} q)(i \text{ad} \sigma)(-i \text{ad} \sigma G_j) \equiv R_{\text{NLS}}(-i \text{ad} \sigma G_j). \)

We call the operator \( R_{\text{NLS}} \) the recursion operator for the NLS hierarchy. The coefficients \( G_j \) are not uniquely determined because of the integral operator \( \partial_z^{-1} \). If we require that \( G = i\sigma \) vanishes at a point, usually at \( z = -\infty \), then they are uniquely determined. We call the matrices \( -i \text{ad} \sigma G_j \) under this condition the principal elements of the hierarchy. Now if \( \tilde{J} = iz^k\sigma \), then \( m \tilde{J} m^{-1} = z^k G \) and the time evolution equation is

\( q_t = i[\sigma, m_{1t}] = -i[\sigma, G_{k+1}] = R_{\text{NLS}}^k q. \)
More generally, if \( \tilde{j} = \sum_{k \geq 0} a_k z^{k-i} \), then

\[
\tilde{q} = G_1 z^{k-1} + G_2 z^{k-2} + \cdots + G_k.
\]

and the integrable evolution equation is

\[
q_t = \sum_{k \geq 1} a_k \text{ad} \sigma R_{\text{NLS}}^k q.
\]

Clearly the coefficients can be any functions in \( t \). We see the amazing fact that the hierarchy is a module with functional (in \( t \)) coefficients. We see in (Q1.2) that the entire hierarchy may be obtained by choosing arbitrary, possibly \( t \)-dependent integration constants (may depend on \( t \)) in (1).

The first few principal elements of the hierarchy are:

\[
\begin{align*}
q_t &= 0, \\
q_t &= -i[\sigma, q], \\
q_t &= -q_z, \\
q_t &= i[\sigma, q_{xx} + 2Q \tilde{Q} q]. \quad \text{NLS} \\
q_t &= q_{xxx} + 2(Q \tilde{Q} q)_x - [q, [q, q_x]]. \quad \text{MKdV}.
\end{align*}
\]

The evolution system (1) can be written as a pair of equations

\[
\begin{align*}
Q_t &= Q_{xxx} + 6Q Q_x \tilde{Q}, \\
\tilde{Q}_t &= Q_{xxx} + 6Q \tilde{Q}_x.
\end{align*}
\]

For \( \tilde{Q} = \pm \tilde{Q} \), the equation reduces to a real one

\[
Q_t = Q_{xxx} \pm 6Q^2 Q_x.
\]

Finally, when \( Q \) is real. we have

\[
Q_t = Q_{xxx} \pm 6Q^2 Q_x.
\]

This equation is refereed as the MKdV equation.

From the first few \( G_j \), one would make a conjecture that the entire hierarchy consists of polynomials of the entries of \( q \) and their derivatives. This is not at all clear from (1) because of the integral operator \( \delta x^{-1} \) in the recursion operator. By the results of Chapter 3 and 4, for any \( q \) of Schwartz class, we have solution \( m \) of equation (1.2.14a) such that \( m \) approaches \( \delta \) rapidly as \( x \to -\infty \) and approaches some diagonal matrix \( \delta(x) \) as \( x \to +\infty \). Hence \( G_j \sigma \) decays rapidly as \( |x| \to \infty \) and the integral of the RHS of (1) is constantly 0 for any \( q \). This implies inductively that the RHS of (1) is always a complete derivative of entries of \( q \) and their derivatives and therefore all \( G_j \) are polynomials of the entries of \( q \) and their derivatives.

On the other hand, the expansion of \( m_{\text{diag}} \) involves with integrals. For example

\[
m_{\text{diag}} = \int \tilde{Q} Q \, dx.
\]
This is because in general $\delta(z) \neq I$. These quantities appear to be very important as they are related to the conservation laws. This will be explained in Chapter 4.

The following are examples of some well known Lax pairs with corresponding integrable equations. Most of these problems are normalized at $z = \infty$, and the normalization point will only be given for the problems not normalized at $\infty$.

**Remark 1.2.15.** We see inductively that the highest order $x$-derivatives in any of the principal element of the hierarchy is linear and the remaining terms contain only derivatives of orders at least 2 less than the leading one.

**Example 1.2.16.** Landau-Lifshitz equation.

\[
S_t = -\frac{i}{2} [S, S_{zz}]
\]

\[S = S_1 \sigma_1 + S_2 \sigma_2 + S_3 \sigma_3,\]

where the vector $(S_1, S_2, S_3)$ lies on the unit sphere, (equivalently, $S^2 = I$). $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices. The Lax pair has the form

\[
\psi_x = \frac{i}{z} S \psi.
\]

\[
\psi_t = \left( \frac{i}{z^2} S + \frac{1}{z} SS_z \right) \psi.
\]

In this problem $J = i \sigma_3 / z$ and $\bar{J} = i \sigma_3 / z^2$.

**Example 1.2.19.** Sine–Gordon equation:

\[
u_{tt} - u_{xx} + \sin u = 0.
\]

The Lax pair is

\[
\psi_x \psi^{-1} = -\frac{i z}{4} \sigma_3 + \frac{i}{4z} \begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 & u_x + u_t \\ u_x + u_t & 0 \end{pmatrix}.
\]

\[
\psi_t \psi^{-1} = -\frac{i z}{4} \sigma_3 - \frac{i}{4z} \begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 & -u_x - u_t \\ -u_x - u_t & 0 \end{pmatrix}.
\]

For this problem, there are two choices of $J$ and $\bar{J}$,

(1.2.22a) $J = -i(z - z^{-1}) \sigma_3 / 4$, $\bar{J} = -i(z + z^{-1}) \sigma_3 / 4$.

(1.2.22b) $J = -i(z + z^{-1}) \sigma_3 / 4$, $\bar{J} = -i(z - z^{-1}) \sigma_3 / 4$.

**Example 1.2.23.** Elliptic sine–Gordon equation:

\[
u_{xx} = -\sin u.
\]

The Lax pair is

\[
\psi_x = \frac{i}{2} (-u_x \sigma_2 + \sigma_3) \psi.
\]
(1.2.25b) \[ \psi_x = \frac{i}{2z} (\sin u \sigma_1 + \cos u \sigma_3) \psi. \]

(1.2.26) \[ J = iz \sigma_3/2, \quad \bar{J} = i \sigma_3/(2z). \]

**Example 1.2.27.** Derivative nonlinear Schrödinger equation:

(1.2.28) \[ iQ_t = -Q_{xx} + \delta(QQ^2)_x, \quad \delta = \pm 1. \]

The Lax pair is

(1.2.29a) \[ \psi_x = -iz^2 \sigma_3 \psi + zq \psi. \]

(1.2.29b) \[ \psi_t = -2iz^4 \sigma_3 \psi + (2z^2q + z^2q_2 + zq_1) \psi. \]

where

\[ q = \begin{pmatrix} 0 & Q \\ \delta Q & 0 \end{pmatrix}, \quad q_1 = \begin{pmatrix} 0 & iQ_z + \delta|Q|^2Q \\ -i\delta Q_x + |Q|^2Q & 0 \end{pmatrix}, \quad q_2 = -i\delta|Q|^2\sigma_3. \]

Finally, we give an example of Lax pair which is related to a vector RH problem instead of a matrix RH problem.

**Example 1.2.30.** KdV equation:

(1.2.31) \[ u_t + uu_{xxx} + 6uu_x = 0. \]

The Lax pair is

(1.2.32a) \[ \psi_x + u \psi = z^2 \psi. \]

(1.2.32b) \[ \psi_t + \psi_{xxx} - 3(z^2 - u)\psi_x - 4iz^2 \psi = 0. \]

The related RH problem is to find a row vector solution with the symmetry \( \psi(z) = \sigma_1 \psi(-z) \sigma_1 \).

In the remaining sections of this chapter, we solve certain RH problems with exact solutions and use these solutions to construct special solution for certain integrable systems.

1.3. **Direct Method.** The direct method is to construct solutions of integrable systems directly from solutions of any RH problems of the form (1.2.2.), in contrast with the inverse scattering method in which the RH problems are first characterized for the class of “potentials” \( q \equiv U - J \) as functions of \( x \) have certain regularities and decays at \( x = \infty \). Throughout this section, we assume that the fundamental solutions \( m \) for the related RH problems exist and that they have asymptotic expansions to sufficient orders at each pole of \( \text{ad} J \) and \( \text{ad} \bar{J} \). This assumption is needed for the smoothness of the RH problem (1.2) in \( x \) and \( \tau \) variables. The solutions of the integrable systems may then be determined by the coefficients of these asymptotic expansions through the equation satisfied by \( m \),

(1.3.1) \[ m_x = (\text{ad} J + q)m. \]

Note that this is just another way of writing the first equation of the Lax pair. What are the sufficient orders of these asymptotic expansions in order to determine \( q \)? This question is answered by Proposition 1.3.2 below. Denote by \( P_J \) the set of poles of \( \text{ad} J \), and by \( m_{J'}(x, z) \) the asymptotic expansion of \( m \) at \( z' \in P_J \).
Proposition 1.3.2. The rational function $q$ can be determined by

\[
(C1.3.3) \quad (\text{ad} J + q)m_{z'} = \begin{cases} 
O(1) & \text{as } z \to z', \ z' \in P_\delta \setminus \{\infty\} \\
O(z^{-1}) & \text{as } z \to \infty, \ z' = \infty.
\end{cases}
\]

(C1.3.3) also gives necessary algebraic conditions for $q$ which barely avoid the appearance of $z$ derivative terms. See (4.3.22.23) for the proof.

Proposition 1.3.4. (Q1.) For $q$ constructed through (C1.3.3), $\text{tr} \ q = 0$.

Example 1.3.5 ZS-AKNS system. $J(z) = zJ_0$. $q = q(x)$. The asymptotic expansion $m_\infty = I + z^{-1}m_{\infty,1} + O(z^{-2})$ is sufficient to determine that

\[
(1.3.6) \quad q = -[J_0, m_\infty,1].
\]

Thus $q$ must satisfy the algebraic condition (see Example 1.2.10 and (Q1.))

\[
(1.3.7) \quad q(x) \in \text{ran ad} \ J_0.
\]

Example 1.3.8. $J(z) = z^{-1}J_0$. $q(x, z) = z^{-1}q_0(x)$. Using $m_0(x, z) = m_{0,0} + O(z)$. we obtain

\[
(1.3.9) \quad J_0 + q = m_{0,0}J_0m_{0,0}^{-1}.
\]

Thus $q_0$ must satisfy the algebraic condition

\[
(1.3.10) \quad J_0 + q_0 \sim J_0.
\]

In Example 1.2.16 $J_0 = i\sigma_2$, $J_0 + i = i\mathcal{S}$, $\sigma_3 \sim \mathcal{S}$.

Example 1.3.11. This example may be viewed as a mixture of the two examples above. $J(z) = zJ_1 + z^{-1}J_2$. $q = q_1 + z^{-1}q_2$. Here $q_1$ and $q_2$ are determined by (1.3.7) and (1.3.9) respectively. Thus

\[
(1.3.12) \quad q_1(x) \in \text{ran ad} \ J_1, \ q_2(x) + J_2 \sim J_2.
\]

These conditions are clearly satisfied in Example 1.2.19.

We remark that similar algebraic conditions are also necessary for the second equation of the Lax pair.

The inverse scattering transforms have been often viewed as certain nonlinear generalization of Fourier transforms which are used for solving linear PDEs. In the following we will make a comparison between the direct method introduced above and the linear superposition method, using the nonlinear Schrödinger equation as an example.

Clearly $(Q, \dot{Q}) \equiv (e^{i\alpha}, e^{-i\alpha})$, $\alpha \equiv z\dot{z} - z^2t$, is a solution of the linear equation

\[
(4.1.13) \quad \begin{cases}
\dot{Q}_t + Q_{zz} = 0 \\
\dot{Q} - \dot{Q}_{zz} = 0.
\end{cases}
\]
By superposition

$$
(4.1.14) \quad \left( \begin{array}{cc} 0 & Q \\ -\tilde{Q} & 0 \end{array} \right) \overset{\text{det ad } \sigma_3}{\underset{4\pi}{\text{ad}}} \int_{\Gamma} e^{i\phi} \text{ad } \sigma_3 w dz
$$

for any \( w = w(z) \) is a solution of (4.1.13).

Consider now the nonlinear system (1.2.13). Assume \( v - I \) is small in \( L^2 \cap L^\infty \). It follows from equation (1.1.20) for \( h = I \). \( \mu - I \) is small in \( L^2 \). Hence in the leading order expansion

$$
(1.3.15) \quad \left( \begin{array}{cc} 0 & Q \\ -\tilde{Q} & 0 \end{array} \right) = -\frac{i}{2} \text{ad } \sigma_3 m_{x,1}
$$

$$
= \frac{\text{ad } \sigma_3}{4\pi} \int_{\Gamma} e^{i\phi} \text{ad } \sigma_3 \mu(v - I) dz - \frac{\text{ad } \sigma_3}{4\pi} \int_{\Gamma} e^{i\phi} \text{ad } \sigma_3 (v - I) dz
$$
gives a solution of the linear problem (1.3.13).

1.4. Rational Problems and Solitons. All meromorphic functions on the Riemann sphere are rational. Rational problems are the RH problems that have rational fundamental solutions. Rational problem can always be reduced to a linear algebraic problem and hence exactly solvable provided the solutions exist. Applying these exact solutions to integrable systems, we obtain exact solutions for integrable systems. This way we obtain all, but not limited to, soliton solutions of integrable systems.

Let \( P \) be a finite subset of \( \mathbb{C} \). Denote by \( V_{z'} \) for \( z' \in P \) the set of \( n \times n \) matrix valued meromorphic germs \( f \) such that either \( f \) or \( f^{-1} \) has a pole at \( z' \). Such a germ is representable by a Laurent series at \( z' \) with a finite order of singularity. An \( r \times n \) matrix function \( m \) is said to be a solution of the RH problem \( (m, v_{z'}, v_{z'} \in V_{z'}, z' \in P) \) if

(1.4.1a) \( m \) is analytic away from \( P \).

(1.4.1b) \( m v_{z'} \) is analytic at \( z' \) for all \( z' \in P \).

Furthermore, an \( n \times n \) matrix function \( m \) is said to be a fundamental solution if in addition to (1.4.1a,b).

(1.4.1c) \( m^{-1} \) is analytic away from \( P \) and \( (mv_{z'})^{-1} \) is analytic at \( z', z' \in P \).

This RH problem can be converted to a standard one by choosing the complete contour \( \Gamma \) as the union of circles centered at \( z' \in P \) with their radii small enough so that within them \( v_{z'}, z' \in P \) have no singularities other than \( z' \), and choosing \( v_{z'} \) as the jump matrix on the circle centered at \( z' \).

A necessary condition for the RH problem described by (1.4.1a,b) to have a fundamental solution is

$$
(1.4.2) \quad \sum_{z' \in P} W_{\Gamma}(\det v_{z'}) = 0,
$$

where \( W_{\Gamma}(\det v_{z'}) \) is the winding number of \( \det v_{z'} \) along the circle centered at \( z' \) mentioned above. By proposition 1.1.4, when (1.4.2) is true, an \( n \times n \) solution \( m \) is fundamental if and only if \( \det m(z) \neq 0 \) for some \( z \), and usually it is easier to check (1.4.2) than to check (1.4.1c).
Clearly, conditions (1.4.1bc) can be written as a finite set of algebraic equations involving the coefficients of the Laurent series of $m$ and $v_2$, and without loss of generality, we may replace $v_2$ by a rational function (Q1.4). The next question is: what type of RH problems can be reduced to rational problems?

Denote

$$M(\tilde{C} \setminus \Gamma) = \{ f : \text{finitely many poles and } L^2 \text{ boundary values} \}$$

and denote by $P_f$ the set of poles of $f$.

**Theorem 1.4.4** If there exists $b \in M$ such that $v = b^{-1}b_{\infty}$, then the RH problem for $(v, \Gamma)$ has a solution in the form $m = M b$ provided $M$ is a solution of the RH problem $(M, b_{\infty}, z' \in P_f \cap P_{b-1})$.

**Proof.** Clearly, $m^0 b$ satisfies $m_{\infty} = m_{\infty} v$ and the analyticity of $m$ follows from the fact that $b \in M$ and that $M b_{\infty}, z' \in P_b$ is analytic at $z'$.

**Remarks 1.4.5.**

1. **(1.4.6a)** The matrix function $m^0 b$ is a fundamental solution for $v$ if and only if $M$ is a fundamental solution for $b_{\infty}$. An $n \times n$ solution $M$ is fundamental if and only if (1.4.2) holds.

2. **(1.4.6b)** The RH problem $(M, b_{\infty}, z' \in P_f \cap P_{b-1})$ may be viewed as a deformation of the original RH problem $(v, \Gamma)$. In general, $\Gamma$ can be deformed within the domain of the holomorphicity of $v$ and $v^{-1}$.

3. **(1.4.6c)** The matrix function $b$ may be viewed as a piecewise meromorphic (instead holomorphic) "solution" of the RH problem. Theorem 1.4 says if one can solve a RH problem modulo some poles, then the problem is rationalized.

4. **(1.4.6d)** When $v$ can be approximated by rational functions: $v = v^{(1)} v^{(2)}$ with $v^{(2)}$ rational and $v^{(1)} - I$ small in $L^\infty$ norm, we can solve the RH problem $m^{(1)}_{\infty} = m^{(1)}_{\infty} v^{(1)}$ through the integral equation $\mu = I + C v^{(a)} \mu$. Hence the problem can be reduced by a rational problem by setting $b_{\infty} = m_{\infty}^{(1)}, b_{\infty} = m_{\infty}^{(1)} v^{(2)}$.

5. **(1.4.6e)** More general type of rational RH problems will be discussed in Chapter 2.

**Example 1.4.7.** Let $P = \{ z_1, z_2 \}$ and $v_{z_1}(z) = \begin{pmatrix} 1 & \frac{z_1}{z-z_1} \\ 0 & 1 \end{pmatrix}$, and $v_{z_2} = \begin{pmatrix} 1 & \frac{z_2}{z-z_2} \\ 0 & 1 \end{pmatrix}$. We look for a solution $m$ normalized at $\infty$. Since det $v = 1$, such a solution must
be fundamental by Proposition 1.1.4. By (1.4.1c), $m$ has two simple poles at $z_1$, $z_2$.

\[ m = I + \frac{A}{z - z_1} + \frac{B}{z - z_2} \]

for some matrices $A$ and $B$. Condition (1.4.1abc) is equivalent to

\[
\begin{align*}
(1.4.9a) \quad & A + (I + \frac{B}{z_1 - z_2}) \begin{pmatrix} 0 & c_1 \\ 0 & 0 \end{pmatrix} = 0, \quad A \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0, \\
(1.4.9b) \quad & B + (I + \frac{A}{z_2 - z_1}) \begin{pmatrix} 0 & 0 \\ c_2 & 0 \end{pmatrix} = 0, \quad B \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0.
\end{align*}
\]

We compute $A_{11} = A_{21} = B_{12} = B_{22} = 0$, $A_{12} = -c_1 / \Delta$, $A_{22} = -B_{11} = c_1 c_2 / ((z_1 - z_2) \Delta)$, and $B_{21} = -c_2 / \Delta$, where $\Delta = 1 + c_1 c_2 / (z_1 - z_2)^2$. Thus this RH problem has a fundamental solution if and only if $\Delta \neq 0$.

**Example 1.4.10—Soliton Solutions.** Let $v = \{v_{11}, v_{22}\}$ be as in Example 1.4.7, and consider the RH problem for $v_{\text{sol}} \overset{\text{def}}{=} e^{i \alpha \sigma_3} v$ where $\alpha = z x - z^2 t$ (Example 1.2.10). We will use the solution of this RH problem to construct single soliton solutions for nonlinear Schrödinger equation. Clearly, $m(z)v_{\alpha_{12}}(z)$ is analytic at $z_j$, $j = 1, 2$ if and only if $m(z)v_{\alpha_{12}}(z)$ is analytic at $z_j$, $j = 1, 2$. Hence we can simply use the solution in Example 1.4.7 by replacing $c_1$ and $c_2$ with $c_1 e^{i \alpha(z_1)}$ and $c_2 e^{i \alpha(z_2)}$, respectively. By (1.4.8),

\[ m = I + z^{-1}(A + B) + O(z^{-2}). \quad \text{as} \quad z \to \infty. \]

From (1.3.6),

\[
\begin{align*}
(1.4.11a) \quad & Q = -iA_{12} = ic_1 e^{i \alpha(z_1)}/(1 + c_1 c_2 e^{i \alpha(z_1) - i \alpha(z_2)}(z_1 - z_2)^{-2}), \\
(1.4.11b) \quad & \dot{Q} = -iB_{21} = ic_2 e^{-i \alpha(z_2)}/(1 + c_1 c_2 e^{i \alpha(z_1) - i \alpha(z_2)}(z_1 - z_2)^{-2}).
\end{align*}
\]

Consider the focusing case, $\dot{Q} = \dot{Q}$. The corresponding symmetry for $m$ is $m^*(\xi)m(z) = I$ (Q1.). Since the complete contour $\Gamma$ is anti-Schwarz reflection invariant ($\Gamma = -\Gamma$), $\psi^*(\xi)\psi(z) = I$. Thus we must have

(1.4.) $z_1 = \bar{z}_2$ and $c_1 = -\bar{c}_2$.

We obtain

\[
(1.4.12) \quad Q = \frac{-ic_1 e^{i \alpha(z_1)}}{1 + \frac{i}{z_1 - z_2} e^{i \alpha(z_1) - i \alpha(z_2)}} = \frac{i c_1}{2} e^{i \xi} e^{i \Re \alpha(z_1)} \sech(\Im \alpha(z_1) + \delta),
\]

where $e^{i \xi} = |(z_1 - z_2)/c_1|$. This solution is called a solitary wave (or single soliton) solution as it is exponentially localized.

Similarly, a solution generated by $n$ pairs of poles in the above type is called an $n$-soliton solution, which behaves like $n$ traveling particles interacting only when they collide each other. The following two figures, generated by Maple, exhibit a solitary wave solution and a 2–soliton solution.
Figure 1.4b. $P = Q$

Figure 1.4c. Re $Q$

The corresponding solution for the linear system (1.3.13) is

\[ Q = \frac{1}{2\pi} \int_{\Gamma} e^{\rho z} v_{12}^z \, dz = ic_1 e^{\alpha z_1}, \]
which grows exponentially in certain directions of $x$ and $t$. This demonstrates an
essential difference between the Fourier analysis and the inverse scattering analysis.
In the case of Fourier analysis for the decaying $Q$, the superposition takes place
on $\mathbb{R}$ where the phase is pure imaginary, while in the case of inverse scattering
analysis takes place not only on $\mathbb{R}$, but also at certain discrete set off $\mathbb{R}$, where
the phase has nonzero real part. In this particular case, some nonlinear cancelation
takes place so that even for exponentially growing RH data, the potential $Q$ decays
exponentially. Soliton solutions represent one of the most interesting nonlinear
phenomena in the theory of PDE. In the defocusing case (Q1.), it is inevitable that
the soliton solutions have singularities for some $x$ and $t$.

1.5. Hyperelliptic Solutions. In this section we deal with the RH problems
with solutions represented by Riemann theta functions which give rise to periodic
and quasi-periodic solutions of integrable systems.

Consider the defocusing nonlinear Schrödinger equation (1.2.13b). We see from
(Q1.) that $m$ normalized at $\infty$ has the symmetry

$$\sigma_1 \overline{m(\overline{z})} \sigma_1 = m(z).$$

Thus $v$ should have the symmetry

$$\sigma_1 v(\overline{z}) \sigma_1 = (v(z))^{-1}$$

for Schwarz reflection invariant contours.

Let $I_j \equiv (a_j, b_j)$, $j = 0, \ldots, N$ a sequence of non-intersecting finite intervals
ordered and oriented from left to right. Set $\Gamma = \bigcup I_j$. Denote $\alpha = x - x^2 t$ and
define

$$(1.5.3) \quad v(x; z) = i e^{\frac{i}{2} \alpha} \sigma_3 e^{i \phi \sigma_3} \sigma_1 = \begin{pmatrix} 0 & i e^{i(\alpha + \phi)} \\ i e^{-i(\alpha + \phi)} & 0 \end{pmatrix},$$

where $\phi = \phi(z)$ is a real valued smooth function on $\Gamma$. Let $g(z)$ be a scalar function
analytic on $\mathbb{C} \setminus \Gamma$ with $L^\infty$ boundary values such that

$$(1.5.4) \quad g_+ + g_- = -\alpha - \phi + \Omega,$$

where $\Omega = \Omega(x, t; z)$ is a piecewise constant function for $z \in \Gamma$ defined as 0 for
$z \in I_0$, and as some functions $\Omega_j = \Omega_j(x, t)$ for $z \in I_j$, $j = 1, \ldots, N$. Functions $\Omega_j$
are to be determined later.

If such a $g$ exists, let

$$(1.5.5) \quad m^{(1)} = me^{-i\phi \sigma_3}.$$

Then $m^{(1)}$ is analytic on $\mathbb{C} \setminus \Gamma$ and

$$(1.5.6) \quad m^{(1)}_+ = m^{(1)}_+ e^{i \phi} \sigma_3 v(x; z) e^{-i \phi} \sigma_3$$

$$= m^{(1)}_+ \begin{pmatrix} 0 & i e^{i(\alpha + \phi + g_+ + g_-)} \\ i e^{-i(\alpha + \phi + g_+ + g_-)} & 0 \end{pmatrix} = m^{(1)}_i e^{i \Omega \sigma_3} \sigma_1.$$

We will see below that the RH problem for the piecewise constant jump function
$\nu^{(1)} \equiv e^{i \Omega \sigma_3} \sigma_1$ can be solved exactly in terms of Riemann theta functions.
To find a $g$, let $p(z) = \prod_{j=0}^{N} (z - a_j)(z - b_j)$, and fix the definition of $\sqrt{p}$ by setting its branch cut to be $\Gamma$ with $\sqrt{p} \sim z^{N-1}$ as $z \to \infty$. It is easy to see (Q2) that

$$g \overset{\text{def}}{=} \sqrt{\frac{-\alpha - \phi + \Omega}{\sqrt{p}_+}}$$

is analytic on $\mathbb{C} \setminus \Gamma$ and satisfies (1.5.4). In order that $g$ be analytic at $\infty$, we need to have $g(z) = O(1)$ and this yields the following moment conditions,

$$\int_{\Gamma} \frac{-\alpha(\tau,z) - \phi(\tau) + \Omega}{\sqrt{p(z)_+}} \cdot z^k \, dz = 0, \quad k = 0, \ldots, N - 1.$$  

Thus we obtain a system of $N$ linear equations for $\Omega_j$, $j = 1, \ldots, N$. This linear system will be solved below.

We denote by $X$ the Riemann surface of $\sqrt{p}$ with two sheets $X_1 \times \mathbb{C} \setminus \Gamma$ glued together along $\Gamma$ in the standard way. Topologically, it is called a torus or a compact Riemann surface of genus $N$. Define $\sqrt{p}$ on $X$ in the way that $\sqrt{p} \sim z^{N+1}$ as $z \to \infty_1 \in X_1$ and $\sqrt{p} \sim -z^{N+1}$ as $z \to \infty_2 \in X_2$.

The following are some basic results in the theory of Riemann surfaces:

Let $\{\alpha_j\}_{j=0}^{N}$ and $\{\beta_j\}_{j=1}^{N}$ denote the cycles indicated in the following figure:

![Figure 1.5](image)

the cycles $\alpha_j$ lie on the first sheet and the cycles $\beta_j$ pass from the first sheet through the slit $J_0$ to the second sheet and back again through $J_0$. The cycles $\{\alpha_j, \beta_j\}_{j=1}^{N}$ form a canonical homology basis for $X$. The dimension of the space of holomorphic differentials is $N$, equal to the genus of $X$. It has a basis $\omega = (\omega_1, \ldots, \omega_N)$ normalized by the condition

$$\int_{\alpha_j} \omega_i = \delta_{ji}.$$  

Furthermore, these differentials have the form $\omega_j = q_j/\sqrt{p}$ with $q_j$ a polynomial of degree less than $N$ with real coefficients. Now we have a simple representation of $\Omega_j$:

$$\Omega_j = 2 \int_{\Gamma} (\alpha + \phi) \omega_j = 2 \int_{\Gamma} \phi \omega_j + \int_{\alpha_k} \omega_j = 2 \int_{\Gamma} \phi \omega_j + 2 \pi i \text{Res}_\infty (\omega_j),$$  

as $\alpha_k$ may be viewed as a positively oriented complete contour enclosing $\infty$. One can also show (Q2) that there exists a unique monomial $q$ of degree $N$ such that

$$\int_{\alpha_j} \frac{q}{\sqrt{p}} \, dz = 0, \quad j = 1, \ldots, N.$$
It then follows from (1.5.7) and (1.5.8) that

\begin{equation}
\begin{aligned}
g(\infty) &= -\frac{1}{2\pi i} \int_{\Gamma} \frac{-\alpha - \phi + \Omega}{\sqrt{\rho}} z^N dz = -\frac{1}{2\pi i} \int_{\Gamma} \frac{-\alpha - \phi + \Omega}{\sqrt{\rho}} q dz \\
&= \frac{1}{2\pi i} \int_{\Gamma} \frac{(\alpha + \phi) q}{\sqrt{\rho}} dz = \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi q}{\sqrt{\rho}} dz - \frac{1}{2} \text{Res}_\infty (aq/\sqrt{\rho}).
\end{aligned}
\end{equation}

Define the Riemann period matrix

\begin{equation}
\tau = (\tau_{ij}) = (\int_{\beta_j} \omega_i)_{1 \leq i,j \leq N}.
\end{equation}

According to [FK], \( \tau \) is symmetric, pure imaginary, and \(-i\tau\) is positive definite. The Riemann theta function is defined as

\begin{equation}
\theta(s) = \sum_{l \in \mathbb{Z}^N} e^{2\pi i (l, s) + \pi i (l, l)} , \quad s \in \mathbb{C}^N.
\end{equation}

It is easy to show that the theta function satisfies

(C1.5.15a) \( \theta(s) = \theta(-s) \),
(C1.5.15b) \( \theta(s + e_j) = \theta(s) \),
(C1.5.15c) \( \theta(s + \tau_j) = e^{\pi i \tau_{ij}} \theta(s) \),

where \( e_j \) is the \( j \)-th column of \( I \) and \( \tau_j = \tau e_j \). We call \( \Lambda = \mathbb{Z}^N + \tau \mathbb{Z}^N \) the period lattice of the theta function.

Denote \( u(z) = \int_{\alpha}^{z} \omega \). Clearly, \( u \) is not a single valued function. Denote

\begin{equation}
M(\cdot, d) = (M_1, M_2) = \left( \frac{\theta(u(z) - \hat{\Omega}/(2\pi) + d)}{\theta(u(z) + d)} , \frac{\theta(-u(z) - \hat{\Omega}/(2\pi) + d)}{\theta(-u(z) + d)} \right),
\end{equation}

where \( \hat{\Omega} = (\Omega_1, \ldots, \Omega_N)^T \) and \( d \) is any vector in \( \mathbb{C}^N \) such that \( \theta(u(z) \pm d) \) do not vanish identically.

**Proposition 1.15.7.** \( M(\cdot, d) \) is meromorphic on \( \mathcal{C} \setminus \Gamma \) and

\begin{equation}
M_+ = M_- \begin{pmatrix} 0 & e^{\Omega} \\ e^{-\Omega} & 0 \end{pmatrix}.
\end{equation}

**Proof.** The multi-valueness of \( u \) is created by the integrals around \( a_j \) cycles. But the periodicity of \( \theta \) makes \( M \) a single valued function on \( \mathcal{C} \setminus \Gamma \). It follows then from the fact that \( \theta(u(z) \pm d) \neq 0 \) that \( M(z, d) \) is meromorphic and the poles of \( M(z, d) \) are precisely the zeroes of \( \theta(u(z) \pm d) \). It is easy to obtain (Q1.) that, for \( z \in I_k \),

\begin{equation}
\begin{aligned}
u_+(z) + \nu_-(z) &= \left\{ \begin{array}{ll}
-\sum_{j=1}^{N} e_j - \tau_k , & k = 1, \ldots, N, \\
0 , & k = 0.
\end{array} \right.
\end{aligned}
\end{equation}

Consequently, we obtain for \( z \in I_k , \ 1 \leq k \leq N \),

\begin{equation}
M_+(z, d) = \left( \frac{\theta(-\nu_-(z) - \sum_{j=1}^{N} e_j - \tau_k - \hat{\Omega}/(2\pi) + d)}{\theta(-\nu_-(z) - \sum_{j=1}^{N} e_j - \tau_k + d)} , \frac{\theta(\nu_-(z) + \sum_{j=1}^{N} e_j + \tau_k - \hat{\Omega}/(2\pi) + d)}{\theta(\nu_-(z) + \sum_{j=1}^{N} e_j + \tau_k + d)} \right)
\end{equation}
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\[
\begin{align*}
&= \left( \frac{\theta(-u_-(z) - \hat{\Omega}(2\pi) + d)e^{2\pi i(-u_- (z) - \Omega_0 / (2\pi) + d_0) - \pi \tau_{ab}}}{\theta(-u_- (z) + d)e^{2\pi i(-u_- (z) + d_0) - \pi \tau_{ab}}} \bigg) \\
&= \left( \frac{\theta(u_-(z) - \hat{\Omega}(2\pi) + d)}{\theta(u_-(z) + d)} \right) \left( \begin{array}{cc} 0 & e^{\Omega_0} \\ e^{\Omega_0} & 0 \end{array} \right) \\
&= M_-(z, d) \left( \begin{array}{cc} 0 & e^{\Omega_0} \\ e^{\Omega_0} & 0 \end{array} \right).
\end{align*}
\]

The proof for the case \( k = 0 \) is similar. \( \square \)

Set \( \gamma(z) \overset{\text{def}}{=} \prod_{j=0}^{N} \left( \frac{z-b_j}{z-a_j} \right)^{1/4} \to 1 \) as \( z \to \infty \), and choose \( \Gamma \) as the branch cut of \( \gamma(z) \). Clearly, \( \gamma_+ = \gamma_-^{1/4} \).

From (Q1.), \( \gamma - \gamma^{-1} \) has precisely \( N \) zeros \( z_1, \ldots, z_N \) located at \( z_j \in (b_{j-1}, a_j), \) \( 1 \leq j \leq N \). We choose

\[
(1.5.22) \quad d = -\sum_{j=1}^{N} \int_{a_j}^{X_2(z_j)} \omega_j,
\]

where \( X_2(z_j) \) is the pre-image of \( z_j \) on the second sheet. According to the theory of Riemann surfaces [FK], \( X_2(z_1), \ldots, X_2(z_N) \) are the \( N \) zeros of \( \theta(u(z) + d) \), while \( \mathcal{X}_1(z_1), \ldots, \mathcal{X}_1(z_N) \) are the \( N \) zeros of \( \theta(u(z) - d) \), using the fact

\[
(1.5.23) \quad -d = \sum_{j=1}^{N} \int_{a_j}^{X_2(z_j)} \omega = -\sum_{j=1}^{N} \int_{a_j}^{X_1(z_j)} \omega \mod \Lambda.
\]

Since \( m^{(2)} \) is defined as on the first sheet \( \mathcal{X}_1 \), it has no singularities on \( \mathcal{C} \setminus \Gamma \). Thus

\[
(1.5.24) \quad m^{(2)}(z) = e^{-ig(\infty)} e^{ig(z)} \prod_{j=0}^{N} (z-b_j) e^{\Omega_0(z)}
\]

is a fundamental solution normalized at \( \infty \). From (1.2.12), \( Q = -i(m_1)_{12} \), where \( m_1 \) is given by the asymptotic expansion \( m = I + z^{-1}m_1 + O(z^{-2}) \), as \( z \to \infty \).

Since

\[
(1.5.25) \quad \gamma - \gamma^{-1} = \frac{1}{2} \sum_{j=0}^{N} (a_j - b_j) + O\left( \frac{1}{z^2} \right), \quad z \to \infty,
\]

we obtain from (1.2),

\[
(1.5.26) \quad Q = \frac{M_2(\infty, d)}{4M_1(\infty, d)} e^{-2i \Omega(\infty)} \sum_{j=0}^{N} (a_j - b_j)
\]

\[
= \frac{i}{4} \sum_{j=0}^{N} (a_j - b_j) \frac{\theta(u(\infty) + d) \theta(\hat{\Omega}(2\pi) + u(\infty) - d)}{\theta(u(\infty) - d) \theta(\hat{\Omega}(2\pi) - u(\infty) - d)} e^{-2i \Omega(\infty)}
\]
1.6 RH Problems Solvable by ODEs. The solutions of linear ODEs are analytic if their coefficients are analytic. These solutions are therefore naturally connected to certain RH problems. Many linear ODEs have been well studied and their solutions are representable by special functions. Thus the connected RH problems have solutions representable by special functions. The problem we study in the following example is the key to the stationary phase type study of oscillatory RH problems for integrable systems [DZ]. Consider the $2 \times 2$ linear ODE

\begin{equation}
\psi_z = -\frac{iz}{2} \sigma_3 \psi + \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \psi.
\end{equation}

We first construct an analytic solution $\psi(z)$ for $\text{Im} \, z > 0$. We start with $\psi_{11}$.

\begin{equation}
\frac{d^2}{dz^2} \psi_{11} = (-\frac{z^2}{4} - \frac{i}{2} + ab) \psi_{11}.
\end{equation}

Setting

\begin{equation}
\psi_{11}(z) = g(e^{-3i\pi/4} z)
\end{equation}

results in the reduction of equation (1.6.2) to the standard parabolic-cylinder equation

\begin{equation}
g_{\zeta \zeta} + \left( 1 - \frac{\zeta^2}{4} + c \right) g(\zeta) = 0.
\end{equation}

where

\begin{equation}
z = iab.
\end{equation}

We choose

\begin{equation}
\psi_{11}(z) = c_1 D_+ (e^{-3i\pi/4} z).
\end{equation}

where $D_+$ denotes the standard (entire) parabolic-cylinder function [WhWa] and $c_1$ some constant, then

\begin{equation}
\psi_{21}(z) = a^{-1} \left( \frac{d \psi_{11}}{dz} + \frac{iz}{2} \psi_{11} \right) = \frac{c_1}{a} \left( \frac{d}{dz} D_+ (e^{-3i\pi/4} z) + \frac{iz}{2} D_+ (e^{-3i\pi/4} z) \right).
\end{equation}

and similarly, we choose

\begin{equation}
\psi_{22}(z) = c_2 D_-(e^{-i\pi/4} z),
\end{equation}

and

\begin{equation}
\psi_{12}(z) = \frac{c_2}{b} \left( \frac{d}{dz} D_- (e^{-i\pi/4} z) - \frac{iz}{2} D_- (e^{-i\pi/4} z) \right).
\end{equation}

Now we define $\psi(z)$ for $\text{Im} \, z < 0$ in a similar manner,
\[ (1.6.10) \quad \psi_{11}(z) = c_3 D_c(e^{i\pi/4} z), \]
\[ (1.6.11) \quad \psi_{21}(z) = \frac{c_3}{a} \left( \frac{d}{dz} D_c(e^{i\pi/4} z) + \frac{iz}{2} D_c(e^{i\pi/4} z) \right). \]
\[ (1.6.12) \quad \psi_{22}(z) = c_4 D_{-c}(e^{3i\pi/4} z), \]
and
\[ (1.6.13) \quad \psi_{12}(z) = \frac{c_4}{b} \left( \frac{d}{dz} D_{-c}(e^{3i\pi/4} z) - \frac{iz}{2} D_{-c}(e^{3i\pi/4} z) \right). \]

From [WhWa], we have as \( \zeta \to \infty \),
\[ (1.6.14) \quad D_c(\zeta) = \begin{cases} 
\zeta^\epsilon e^{-\frac{\zeta^2}{4T}} (1 + O(\zeta^{-2})) & \text{if } |\arg \zeta| < \frac{3\pi}{4}, \\
\zeta^\epsilon e^{-\frac{\zeta^2}{4T}} (1 + O(\zeta^{-2})) - \frac{\sqrt{2\pi}}{\Gamma(-c)} e^{-\pi i} \zeta^{-\epsilon-1} e^{\frac{\zeta^2}{4T}} (1 + O(\zeta^{-2})) & \text{if } \frac{\pi}{4} < |\arg \zeta| < \frac{3\pi}{4}, \\
\zeta^\epsilon e^{-\frac{\zeta^2}{4T}} (1 + O(\zeta^{-2})) - \frac{\sqrt{2\pi}}{\Gamma(-c)} e^{-\pi i} \zeta^{-\epsilon-1} e^{\frac{\zeta^2}{4T}} (1 + O(\zeta^{-2})) & \text{if } -\frac{\pi}{4} < |\arg \zeta| < \frac{\pi}{4},
\end{cases} \]

where \( \Gamma \) is the Gamma function. Set
\[ (1.6.15) \quad H = \psi z^{-c\sigma_3} e^{iz^2/4\sigma_3}. \]

We want to find \( c_j, j = 1, 2, 3, 4 \) to normalize \( H(z) \) for \( \pm \text{Im } z > 0 \) at \( z \) along the ray \( e^{i\pi \zeta} \) for some small \( \epsilon \). We choose
\[ (1.6.16) \quad c_1 = c_4 = e^{3i\pi c/4}, \text{ and } c_2 = c_3 = e^{-i\pi c/4}. \]

By the asymptotics given in (1.6.14), it is easy to verify that \( H \) is normalized as we wanted. In fact \( H \) is normalized at \( \infty \) along any curve nontangential to \( \mathcal{R} \). Since
\[ (1.6.17) \quad \text{det } H = \text{det } \psi \text{ is piecewise constant in } \mathbb{C} \setminus \mathbb{R}. \]

By the normalization condition \( \text{det } H = \psi = 1 \).

Now we are ready to compute the constant jump matrix \( \psi = \psi_+^{-1} \psi_-. \)

Letting \( z \to \infty \) along \( \mathbb{R}_+ \), we obtain
\[ (1.6.18) \quad \nu_{22} = c_2 c_3 e^{i\pi c/2} = 1. \]

We also compute
\[ (1.6.19) \quad \nu_{12} = \psi_{22} - \psi_{12}^{2+} - \psi_{12} - \psi_{22}^{2+} = c_4 c_1 \omega(D_c(e^{3i\pi/4} z), D_{-c}(e^{-i\pi/4} z)) = -\frac{\sqrt{2\pi e^{\frac{i\pi c}{4}} - i\pi}}{b \Gamma(c)}. \]
\begin{equation}
\psi_{21} = \psi_{21} - \psi_{11} - \psi_{11} - \psi_{21} + \psi_{12}
= \frac{c_1 c_2}{a} W(D_c(e^{i\pi/4}z), D_c(e^{-3i\pi/4}z)) = \frac{\sqrt{2}\pi e^{i2\pi c - i\frac{\pi}{4}}}{a\Gamma(-c)}.
\end{equation}

and

\begin{equation}
v_{11} = 1 + v_{12} v_{21} = 1 + \frac{-2\pi e^{i\pi c}}{ab\Gamma(c)\Gamma(-c)} = 1 + 2ie^{i\pi c} \sin \pi c = e^{2i\pi c}.
\end{equation}

See [AbSt], p.687 for the calculation of the Wronskians.

Assume $|Re c| < 1/2$, then $m \overset{\text{def}}{=} \psi e^{iz^2/4\sigma_3} \in \mathbb{E}^2$ and is a fundamental solution of the RH problem for

\begin{equation}
(e^{-\frac{i\sigma_3^2}{4}} \overset{\text{ad}}{\sigma_3} \psi, \mathbb{R}).
\end{equation}

In the classical stationary phase method, oscillatory integrals are canonically reduced to the form $e^{-iz^2}$ after certain translations and dilations, while (1.6.22) is the canonical reduced form for oscillatory RH problems.

It readily verifies that

\begin{equation}
m^{(1)}(z) \overset{\text{def}}{=} e^{-\frac{i\sigma_3}{4}} \overset{\text{ad}}{\sigma_3} m(\sqrt{2t}(z - x)/(2t))
\end{equation}

is a fundamental solution of the RH problem

\begin{equation}
(e^{\frac{1}{2}(z-x)^2} \overset{\text{ad}}{\sigma_3} \psi, \mathbb{R}).
\end{equation}

Thus we can construct a solution of the nonlinear Schrödinger equation through $m^{(1)}$. We compute the large $|z|$ asymptotics

\begin{equation}
(\sqrt{2t})^{-c\sigma_3} m^{(1)}(z) = \left( I + \frac{e^{-\frac{i\pi}{4} + \frac{i\sigma_3}{4} \log(2t)} \overset{\text{ad}}{\sigma_3}}{\sqrt{2t}(z - \frac{x}{2t})} + O(\frac{1}{|z|^2}) \right) \left( z - \frac{x}{2t} \right)^{c\sigma_3}.
\end{equation}

Substituting this asymptotic form into (1.3.1), we obtain

\begin{equation}
\begin{pmatrix}
0 & Q
\end{pmatrix}
= -e^{-i(\frac{\sigma_3^2}{4} - \frac{1}{2} \log(2t)) \overset{\text{ad}}{\sigma_3}} \frac{i \overset{\text{ad}}{\sigma_3}}{2\sqrt{2t}} H_1.
\end{equation}

Substituting

\begin{equation}
\psi = Hz^{c\sigma_3} e^{-iz^2/4\sigma_3} = (I + z^{-1} H_1 + O(z^{-2})) z^{c\sigma_3} e^{-iz^2/4\sigma_3}
\end{equation}

into (1.6.1), we obtain

\begin{equation}
\begin{pmatrix}
0 & a
-\bar{Q} & 0
\end{pmatrix}
= \frac{i}{2} [\sigma_3, H_1].
\end{equation}

It then follows

\begin{equation}
\begin{pmatrix}
0 & a
-\bar{Q} & 0
\end{pmatrix}
= -e^{-i(\frac{\sigma_3^2}{4} - \frac{1}{2} \log(2t)) \overset{\text{ad}}{\sigma_3}} \frac{i \overset{\text{ad}}{\sigma_3}}{2\sqrt{2t}} \begin{pmatrix}
0 & a
b & 0
\end{pmatrix}.
\end{equation}
\[ (1.6.30) \quad \left( \begin{array}{c}
0 \\
-e^{-i \pi} \frac{z^c - c \log(2t)}{\sqrt{1 - c^2 \frac{t}{r_{12}}}} \\
\frac{\sqrt{1 - c^2 \frac{t}{r_{12}}} \Gamma(-c)}{\sqrt{1 - c^2 \frac{t}{r_{12}}} \Gamma(-c)}
\end{array} \right) \]

where \( c \) can be computed as follows.

\[ (1.6.31) \quad c = iab = \frac{-2i\pi c e^{i\pi c}}{v_{12}v_{21}} \frac{\Gamma(c)\Gamma(-c)}{1 + v_{12}v_{21}} = \frac{c(e^{2i\pi c} - 1)}{v_{12}v_{21}}. \]

Hence

\[ (1.6.32) \quad c = \frac{1}{2\pi i} \log(1 + v_{12}v_{21}) = \frac{1}{2\pi i} \log v_{11}. \]

**Remark 1.6.33.** If we remove the log \( t \) term from expression (1.6.30), we obtain a solution of the linear Schrödinger equation.

Consider the Bessel equation

\[ (1.6.34) \quad z^2 y'' + zy' + (z^2 - \nu^2) y = 0. \]

where \( \nu \) is a parameter. A fundamental set of solutions of the Bessel equation are the so-called Hankel functions \( H^{(1)}_{\nu}(z) \) and \( H^{(2)}_{\nu}(z) \) with the following relations

\[ (1.6.35a) \quad H^{(1)}_{\nu}(ze^{i\pi}) = -e^{-i\nu\pi} H^{(2)}_{\nu}(z) \]
\[ (1.6.35b) \quad H^{(2)}_{\nu}(ze^{i\pi}) = e^{i\nu\pi} H^{(1)}_{\nu}(z) + 2\cos \nu\pi H^{(2)}_{\nu}(z). \]

Thus the matrix function

\[ (1.6.36) \quad \psi_{\nu}^{(1)}(z) = \psi_{\nu}^{-}(z) \left( \begin{array}{cc}
0 & e^{i\nu\pi} \\
-e^{-i\nu\pi} & 2\cos \nu\pi
\end{array} \right) \]

is analytic for \( z \in \mathbb{C} \setminus \{-\infty, 0\} \) and satisfies the jump condition

\[ (1.6.37) \quad \psi_{\nu}^{-}(z) = \psi_{\nu}^{+}(z), \quad z \in (-\infty, 0). \]

Set

\[ (1.6.38) \quad m = \frac{\sqrt{\pi} e^{i\pi/4}}{2} \left( \begin{array}{cc}
1 & 0 \\
-1 & -i
\end{array} \right) \psi e^{-i(\sqrt{z} + \sqrt{z})} \sigma_3, \]

we obtain

\[ (1.6.39) \quad m_{+}(z) = m_{-}(z) \left( \begin{array}{cc}
0 & 1 \\
-1 & 2\cos \pi\nu e^{-2i(\sqrt{z})}
\end{array} \right), \quad z \in (-\infty, 0). \]

Using the asymptotics of the Hankel functions as \( z \to \infty \),

\[ (1.6.40) \quad H^{(1)}_{\nu}(z) = \sqrt{\frac{2}{\pi z}} e^{i(\nu - \frac{\pi}{4})} \left( 1 + \sum_{n=1}^{\infty} \frac{\left( \frac{1}{2} + \nu \right) n\left( \frac{1}{2} - \nu \right)}{n!(2iz)^n} \right), \quad -\pi < \arg z < 2\pi. \]
32 \[ H_{\nu}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{-\pi iz - \frac{\pi z}{2}} \left( 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(\frac{1}{2} + \nu)_n (\frac{1}{2} - \nu)_n}{n!(2iz)^n} \right), \quad -2\pi < \arg z < \pi. \]

where \((a)_n = a(a+1) \ldots (a+n-1)\), we obtain the asymptotics

\[ m = \frac{1}{\sqrt{2}} \begin{pmatrix} z^{-1/4} & -z^{-1/4} \\ z^{1/4} & z^{1/4} \end{pmatrix} \left( 1 + O\left(\frac{1}{\sqrt{z}}\right) \right), \quad z \to \infty. \]

We conclude that \(m\) is a fundamental solution of the RH problem (1.6.39).

Now consider the Airy equation

\[ y'' - zy = 0. \]

The Airy function \(Ai(z)\) is a solution of the Airy equation with asymptotics as \(z \to \infty, |\arg z| < \pi\).

\[ Ai(z) = \frac{z^{-1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}z^{3/2}} (1 + O(|z|^{-3/2})). \]

(1.6.45)

\[ Ai'(z) = \frac{-z^{1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}z^{3/2}} (1 + O(|z|^{-3/2})). \]

where (1.6.45) can be derived from (1.6.44) through differentiation. Denote by \(\omega = e^{i2\pi/3}\) the cubic root of unity. Clearly from (1.6.43), together with \(Ai(z)\), \(Ai(\omega z)\) and \(Ai(\omega^2 z)\) are all solutions of the Airy equation and we can take any two of them as a fundamental set of solutions. Using the relation

\[ ( Ai(z), Ai(\omega^2 z) ) = ( Ai(z), -\omega^2 Ai(\omega z) ) \begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix}, \]

we obtain

\[ (Ai(z), Ai(\omega^2 z)) = (Ai(z), Ai(\omega^2 z)) \begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix}. \]

Thus the matrix function

\[ \psi(z) \text{ def} = \begin{pmatrix} Ai(z) & Ai(\omega^2 z) \\ Ai'(z) & \omega^2 Ai'(\omega^2 z) \end{pmatrix}, \quad \text{Im} z > 0, \]

\[ \begin{pmatrix} Ai(z) & -\omega Ai(\omega^2 z) \\ Ai'(z) & - Ai'(\omega z) \end{pmatrix}, \quad \text{Im} z < 0 \]

satisfies the jump condition

\[ \psi_+(z) = \psi_-(z) \begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{R}. \]

For the asymptotics \(Ai(\omega z), -\pi \leq \arg z \leq 0\), we have by (1.6.45, 46)

\[ Ai(\omega z) = \frac{e^{-i\pi/6} e^{-1/4}}{2\sqrt{\pi}} e^{\frac{2}{3}z^{3/2}} (1 + O(|z|^{-3/2})). \]
(1.6.51) \[ \text{Ai}'(\omega z) = -\frac{e^{i\pi/6}z^{1/4}}{2\sqrt{\pi}} \epsilon^{\frac{3}{2}z^{3/2}}(1 + O(|z|^{-3/2})). \]

For the asymptotics \( \text{Ai}(\omega^2 z) \), \( 0 \leq \arg z \leq \pi \), we have by (1.6.45.46)

(1.6.52) \[ \text{Ai}(\omega^2 z) = \text{Ai}(\omega^2 e^{-i2\pi} z) = \frac{e^{i\pi/6}z^{-1/4}}{2\sqrt{\pi}} \epsilon^{\frac{3}{2}z^{3/2}}(1 + O(|z|^{-3/2})). \]

(1.6.53) \[ \text{Ai}'(\omega^2 z) = \text{Ai}'(\omega^2 e^{-i2\pi} z) = \frac{-e^{-i\pi/6}z^{1/4}}{2\sqrt{\pi}} \epsilon^{\frac{3}{2}z^{3/2}}(1 + O(|z|^{-3/2})). \]

Set

(1.6.54) \[ m = \sqrt{2\pi}e^{-i7\pi/12} \epsilon e^{\frac{3}{2}z^{3/2} - \frac{1}{2}i\pi} \sigma_3. \]

One verifies

(1.6.55) \[ m_m^{-1} = \begin{cases} \begin{pmatrix} e^{\frac{3}{2}z^{3/2}} & 1 \\ 0 & e^{-\frac{3}{2}z^{3/2}} \end{pmatrix} & z \in \mathbb{R}_- \\ \begin{pmatrix} 1 & e^{-\frac{3}{2}z^{3/2}} \\ 0 & 1 \end{pmatrix} & z \in \mathbb{R}_+ \end{cases}. \]

and

(1.6.56) \[ m(z) = \frac{1}{\sqrt{2i}} \begin{pmatrix} z^{1/4} & iz^{-1/4} \\ -z^{1/4} & iz^{1/4} \end{pmatrix}(1 + O(\frac{1}{|z|^{3/2}})). \quad z \to \infty. \]

where the asymptotics is valid (uniformly) for the first column when \( |\arg z| \leq \pi - \epsilon \), and for the second column when \( -\pi \leq \arg z \leq \pi \). One may use the relation (1.6.47) to compute the uniform asymptotics for the first column when \( \arg z \) is near \( \pm \pi \). Some additional bounded terms will appear in the asymptotics. Thus \( m \) is an \( L^2 \) fundamental solution by Proposition (1.1.4).

**Exercises.**

(Q.1.) Show that the Mobius transforms of \( \mathcal{C} \) give rise to equivalent \( L^p \) norms.

(Q.2.) Position \( S^2 \) with equator on \( S^1 \) of \( \mathbb{C} \). Clearly, for any point \( Z \in S^2 \), other than the north pole \( N(\infty) \), the projection from \( N \) through \( Z \) reaches \( \mathcal{C} \) at precisely one point \( z \). This projection from \( S^2 \) \( \setminus \) \( N \) to \( \mathbb{C} \) is the stereographic projection. Show that the standard metric \( |dZ| \) on \( S^2 \) is projected to \( 2|dz|/(1+|z|^2) \) on \( \mathbb{C} \).

(Q.1.) Let \( S^1 \) be the counterclockwise oriented unit circle. Show that the RH problem for \( (z, S^1) \) does not have a fundamental solution. (Hint: )

What if the complete contour \( S^1 \) above is replaced by \( 2 + S^1 \)?

(Q.1.) Show that \( m(z) \) defined as \( e^{iz} \) for \( \text{Im } z > 0 \) and as \( 1 \) for \( \text{Im } z < 0 \) is a solution of the RH problem for \( (e^{iz}, \mathbb{R}) \) but not a fundamental solution.

Hint: Saying that \( m \) has \( L^2 \) boundary values on \( \mathbb{R} \) is different from saying that \( m \) has boundary values \( m_\pm \) a.e. so that \( m_\pm \in L^2 \). It actually means that \( m \) restricted on parallel lines \( \text{i} \epsilon + \mathbb{R} \) has \( L^2 \) limits as \( \pm \epsilon \searrow 0 \).

Show that this RH problem does not have a fundamental solution.
(Q1.) Let $A$ be an $n \times n$ matrix and define $\text{ad} A = [A, \cdot]$ as an operator on the $n \times n$ matrix algebra. Show that $e^{\text{ad} A} B = e^A B e^{-A}$.

Hint: Consider a differential equation satisfied by $H(t) = e^{t \text{ad} A} B$.

(Q1.) Show that the following statements are equivalent:

a. The contour $\Gamma$ can be reoriented into a complete contour.

b. The two color problem for the map $\mathcal{C} \setminus \Gamma$ is solvable, that is, we can color each connected component of $\mathcal{C} \setminus \Gamma$ by either the “+” color or the “−” color so that the components with the same color do not share (one dimensional) borders.

c. If we stand at any location on $\Gamma$, we find that there are an even number of branches extending out from that location.

How many, if any, different orientations are there in order to make $\Gamma$ into a complete contour?

(Q1.) Show that we can always add some new curves into $\Gamma$ so that it can be made into a complete contour $\hat{\Gamma}$ with certain orientations. Thus if we have a RH problem on $\Gamma$, we may redefine an equivalent RH problem on $\hat{\Gamma}$ by

a. extending $\nu$ to the newly added curves as the identity matrix $I$, and

b. replacing $\nu$ with $\nu^{-1}$ where $\Gamma$ and $\hat{\Gamma}$ have different orientations.

(Q1.) Show that $m = e^{C_0}$ is a fundamental solution of the RH problem for $(e^0, \Gamma)$ as long as both $e^{C_0}$ have $L^2$ boundary values. Here $C$ stands for the Cauchy operator defined in §1.

(Q1.) Consider the special case when $\Gamma = \mathbb{R}$. Show that

a. $C^\pm$ is a pseudodifferential operator with symbol $\chi_{\mathbb{R}}(\mp id/dz)$, where $\chi$ is the indicator function and $\mathbb{R}^\pm$ denotes the positive (negative) real axis. In other words, $f \mapsto C f$ is a convolution operator.

b. any $f \in L^2$. $(C^\pm f)(\zeta) = \pm \chi_{\mathbb{R}^\pm}(\zeta) \hat{f}(\zeta)$. Hint: Show first that the statement is true for any “test” function $f$ which is smooth with compact support.

c. Use a. and the fact that the Fourier transform $\hat{\cdot}$ is unitary on $L^2$, to deduce that $C^-$ and $C^-$ are orthogonal complementary projections.

We remark that these projections are orthogonal only when $\Gamma$ is either a line or a circle.

(Q1.) Assume that $f \in L^2(\Gamma)$. Show that $m = \begin{pmatrix} 1 & Cf \\ 0 & 1 \end{pmatrix}$ is a fundamental solution of the RH problem for $(\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}, \Gamma)$.

We remark that if the function $f \in L^\infty$ is not in $L^2$, we can always consider a new RH problem through a conformal automorphism so that $\infty \not\in \Gamma$.

(Q1.) Let $\Gamma = S^1$, the counterclockwise oriented circle, and $f = \sum_{k=2}^{\infty} \frac{z^k - z^{-k}}{k \ln k}$.

Show that

a. The function $f$ is continuous on $\Gamma$.

b. $(C^\pm f)(z) = \sum_{k=2}^{\infty} \frac{z^k}{k \ln k}$.

c. Show that $C^\pm f$ are not continuous.

(Q1.) Let $f$ be continuous and in $L^2$. Show that for a smooth complete contour $\Gamma$ if $C^\pm f$ is continuous, then $Cf$ tends to its boundary values continuously. Hint: Use the identity $C^+ - C^- = 1$ and the Poisson integral.

Note that since the proof of this result is local, we only need to know that $f$ is continuous somewhere locally to conclude that $Cf$ tends to its boundary values there locally. Besides, $\Gamma$ does not even have to be a countour.
(Q1.) Let $\Gamma = \mathbb{R}$ and denote by $Z_r$ the multiplier $|z|^r$. Show that the operators $Z_r^{-1} C^\pm Z_r$ are bounded from $L^2 \to L^2$ for $-1/2 < r < 1/2$.

Hint: Show that $\| (Z_r^{-1} C^\pm Z_r f, g)_{L^2} \| \leq c \| f \|_{L^2} \| g \|_{L^2}$ for some $c > 0$, using polar coordinates.

(Q1.) Show that $H^1$ is contained in the Hölder space $H^{1/2}$.

(Q1.) Let $\Gamma$ be a smooth complete contour. Show that the following statements are equivalent:

(i). $f \in H^1(\Gamma)$, i.e. $f, f' \in L^2$ in distribution sense.

(ii). $f \in L^2$ and is absolutely continuous (hence $f'$ is defined a.e.). Also $f' \in L^2$.

(iii). There is a sequence of $C^\infty$ functions $f_n$ with compact support. Cauchy in $H^1$ norm (hence in $L^2$ norm), with their $L^2$ limit equal to $f$.

(Q1.) Show that the set of $L^2(\Gamma)$ rational functions is dense in $H^1$. Is the set also dense in the Hölder spaces?

(Q1.) Assume that $v$ is Schwarz reflection invariant in the sense

(i). $\Gamma$ (with its orientation) is invariant under reflection $z \mapsto \bar{z}$, i.e. $\bar{\Gamma} = \Gamma$.

(ii). $v(z) = v(\bar{z})^*$. Also assume that a fundamental solution $m$ for $v$ exists. Show $m(z)m(\bar{z})^* = I$.

Find similar results for inversion $z \mapsto 1/z$ invariant $v$.

We remark it for Schwarz reflection or inversion invariant $v$, there always exist fundamental solutions (see Chapter 2).

(Q1.) Consider an $(n+1) \times (n+1)$ problem for a generalized nonlinear Schrödinger equation. Let the first equation of $\phi$, Lax pair be of the same form as (1.2) with $\delta_3$ replaced by $\delta_3 \equiv \text{diag}[1, \ldots, 1, -1]$. $Q$ by $Q^{\text{def}}(Q_1, \ldots, Q_n)^T$, and $\dot{Q}$ by $(\dot{Q}^{\text{def}}(\dot{Q}_1, \ldots, \dot{Q}_n))$. Find a second equation for the Lax pair similar to (1.2) and the corresponding integrable system as a generalization of (1.2). This generalized system is called a vector nonlinear Schrödinger equation.

(Q1.) A single soliton solution $f$: $n^\prime$ linear Schrödinger equation.

CHAPTER 2. GOHBERG-KREIN FACTORIZATION PROBLEMS AND RELATED SINGULAR INTEGRAL OPERATORS

The Gohberg-Krein (GK) factorization can be viewed as a natural extension of the RH problems. In this Chapter, we study the relation between the GK factorization problem and a class of Fredholm singular integral operators.

2.1. Fredholm Operators. Consider an operator $A \in \mathcal{B}$ for some Banach space $\mathcal{B}$. The operator $A$ is said to be Fredholm if

(C2.1.1a) $\dim \ker A < \infty$.

(C2.1.1b) $\dim \text{coker } A^{\text{def}} \dim \mathcal{B} / \text{ran } A < \infty$.

Note that in general for a bounded operator $A$, ran $A$ need not be closed, hence the quotient $\mathcal{B} / \text{ran } A$ is defined in linear algebraic sense. However, one can prove that ran $A$ is closed provided $\mathcal{B} / \text{ran } A < \infty$ [ref]. When $A$ is Fredholm, we call the integer $\kappa(A)^{\text{def}} = \dim \ker A - \dim \text{coker } A$ the Fredholm index. When $\kappa = 0$, we have the so-called Fredholm alternative: $A$ is invertible if and only if either $\ker A = 0$ or $\text{coker } A = 0$.

The following are some basic results regarding Fredholm operators.

**Proposition 2.1.2.** $A$ is Fredholm if and only if there exist $L, R \in \mathcal{B}$ such that both $LA - 1$ and $AR - 1$ are compact.
$L$ and $R$ are called the left and right regulators of $A$ respectively. Since the
subalgebra $C_{om}$, consisting of all compact operators, is a closed two-sided ideal
in $B \otimes$, the quotient algebra $B \otimes /C_{om}$, which is called the Calkin algebra, is a
Banach algebra. The norm

$$
(2.1.3) \quad \|A\|_{B \otimes /C_{om}} \overset{\text{def}}{=} \inf_{B \in C_{om}} \|A + B\|_{B \otimes}
$$

Proposition (2.1.2) says that the multiplicative group of the Calkin algebra consists
precisely the classes represented by Fredholm operators. Since $L = R \mod C_{om}$,
we may choose $L = R$. It is easy to see now

**Proposition 2.1.4.** The index $\kappa$ is a continuous function from the space of Fredholm
operators to $\mathbb{Z}$. Hence if $t \mapsto A(t)$ is a continuous map from a connected topological
space to the space of Fredholm operators, then the Fredholm index $\kappa(A(t))$ is constant.

Some useful properties of Fredholm operators are

(2.1.5) If $A$ is Fredholm and $E$ is compact, then $A + E$ is also Fredholm with
$\kappa(A + E) = \kappa(A)$. Hence $\kappa$ is also well defined on the Calkin algebra with the norm.
In addition, if $B$ is also Fredholm, $\kappa(AB) = \kappa(A) + \kappa(B)$.

**Lemma 2.1.6.** If $\Gamma$ is a complete contour without self-intersections and if $w$ is
a continuous matrix function on $\Gamma$. then $[C^\pm, w]$ and $C^\pm(C^\pm \otimes w)$ are compact in
$L^2(\Gamma \otimes C^\pm)$.

For complete contours with self-intersections, the “continuity” of $w$ has a dif-
ferent meaning which will be discussed in §2.3, and this Lemma will be generalized.

**Theorem 2.1.7.** Let again $C_v = C^{-} \otimes (v - I)$. then

$$
(2.1) \quad (1 - C_v)(1 - C_{v^{-1}}) = 1 + C^{-}(C^+ \otimes (v^{-1} - I))(v - I) \equiv 1 + T_v.
$$

If $\Gamma$ is a complete contour without self-intersections and if $v$ is continuous on
$\Gamma$, then $1 - C_v$ is Fredholm with the two-sided regulator $1 - C_{v^{-1}}$ in the $L^p$ space.
$1 < p < \infty$.

The above results also hold when $C^\pm$ is replaced by $C^\pm_{\Gamma, \alpha}$ in the space $L^p$. $1 <
p < \infty$.

**Proof.** If $\Gamma$ is a complete contour without self-intersections, then $T_v$ and $T_{v^{-1}}$ are
compact by Lemma 2.1. Thus we only need to prove the identity (2.1). Using the
fact that $\pm C^\pm$ are complementary projections, we have

$$
C_v C_{v^{-1}} = C^{-}(C^+ \otimes (v^{-1} - I))(v - I)
= C^{-}(C^+ - 1) \otimes (v^{-1} - I))(v - I)
= C^{-}(C^+ \otimes (v^{-1} - I))(v - I) - C^{-} \otimes (v^{-1} - I)(v - I)
= C^{-}(C^+ \otimes (v^{-1} - I))(v - I) + C_v + C_{v^{-1}}.
$$

The case of $C^\pm_{\Gamma, \alpha}$ may be proved in the same manner. \Box

The following theorem expresses explicitly the operator $(1 - C_v)^{-1}$ in terms of
a fundamental solution $m$ of the RH problem.
2.2 Gohberg–Krein Matrix Factorization Problems. As discussed in Chapter 1, without loss of generality, we assume that \( \infty \notin \Gamma \) so that \( L^p = L^p \). If this is not convenient in any applications, we may always replace \( L^p \) by \( L^p \) and replace the Cauchy integral operators by their \( L^2 \) extensions. The entire theory developed in this Chapter remains true.

We also assume throughout this Chapter that \( \Gamma \) is a complete contour (Q.1.1). To understand the problem, let us start with the scalar problem. Assume that \( \Gamma \) be a smooth complete contour and \( v \in H^1 \). We see in §1.2 that the RH problem is solvable if and only if \( W_\Gamma v = 0 \). And if this is true, \( m \overset{\text{def}}{=} e^{C \log v} \) is a fundamental solution.

In general \( W_\Gamma (v) = \kappa \in \mathbb{Z} \). \( W_\Gamma (v z^{-\kappa}) = 0 \). Thus

\[
(2.2.1) \quad v = m^{-1} z^\kappa m^+.
\]

where \( m \overset{\text{def}}{=} e^{C \log (v z^{-\kappa})} \). Formula (2.1.1) is called a GK factorization of \( v \) and \( \kappa \) is called the index of \( v \).

Now let us define the \( L^p \) theory of GK factorizations for \( n \times n \) problems.

**Definition 2.2.2.** Let \( \Gamma \) be a complete contour and fix \( z_{\pm} \in \Omega_{\pm} \) and \( 1 < p < \infty \). A jump matrix \( v \) is said to have an \( L^p \) factorization if there exists a matrix function \( m \) on \( \mathbb{C} \setminus \Gamma \) such that

\[
(2.2.3a) \quad m \in L^p \quad \text{and} \quad m^{-1} \in L^{p'}, \quad 1/p + 1/p' = 1.
\]

\[
(2.2.3b) \quad v = m^{-1}(z) D(z) m(z)^+ \quad \text{where} \quad D(z) = \text{diag}(\frac{z - z_+}{z - z_-}, \ldots, \frac{z - z_+}{z - z_-})^\kappa
\]

for some integers \( \kappa_1, \ldots, \kappa_n \). The factorization is said to be a \( \Phi \)-factorization if in addition

\[
(2.2.3c) \quad \text{The operator } f \mapsto (C \Phi f m^{-1}) m_+ \text{ is bounded on } L^p.
\]

The integers \( \kappa_1, \ldots, \kappa_n \) are uniquely determined by \( v \) up to \( \Phi \) order, and we will call them the partial indices of \( v \). The sum of them \( \kappa = \kappa(v) \) will be called the total index of \( v \).

In the case \( 0 \in \Omega_{+} \) and \( \infty \in \Omega_{-} \), the factorization may be written in a simpler form

\[
(2.2.4) \quad v = m^{-1} \text{diag}(z^{\kappa_1}, \ldots, z^{\kappa_n}) m_+.
\]

The fundamental theorem of GK factorization is

**Theorem 2.2.5.** The matrix function \( v \) on a complete contour \( \Gamma \) admits an \( L^p \) factorization if and only if the operator \( 1 - C_v \) applied on \( r \times n \) matrix functions, is Fredholm and

\[
(2.2.6) \quad \dim \ker (1 - C_v) = -r \sum_{\kappa_j < 0} \kappa_j, \quad \dim \ker (1 - C_v) = r \sum_{\kappa_j > 0} \kappa_j,
\]

and therefore

\[
(2.2.7) \quad \kappa(1 - C_v) = -r \kappa(v) = -r(\kappa_1 + \cdots + \kappa_n).
\]

As a corollary of Theorems 2.1.7 and 2.2.5 we have
Theorem 2.2.8. Every continuous $v$ on a complete contour without self-intersections admits an $L^p$ factorization for any $1 < p < \infty$.

It follows directly from the uniqueness of factorizations that

Proposition 2.2.9. If $v$ is a continuous scalar function on a complete contour $\Gamma$ without self-intersections, then

$$\kappa(v) = W_T(\det v).$$

In practice, the contour $\Gamma$ is often not a complete contour. Although $\Gamma$ can always be augmented to a complete contour, the form (C2.2.3b) is not invariant under augmentation unless all the partial indices are zero. In the following, we will slightly alter the form of the GK factorization so that it is not only invariant under the augmentation of the complete contour but also fits an arbitrary complete contour.

Definition 2.2.10. Fix $z_0 \not\in \Gamma$ and $1 < p < \infty$. A jump matrix $v$ is said to have an $L^p$ factorization if there exists a matrix function $M$ such that

(C2.2.11a) $M$ and $M^{-1}$ are analytic on $\mathbb{C} \setminus (\Gamma \cup \{z_0\})$, with $L^p$ and $L^p.1/p+1/p' = 1$ boundary values respectively.

(C2.2.11b) $v = M^{-1}M_+$ and $\text{diag}((z - z_0)^{-\kappa_1} \ldots (z - z_0)^{-\kappa_n})M$ is analytic and nonsingular (as a matrix) at $z_0$ for some integers $\kappa_1, \ldots, \kappa_n$.

A GK factorization is said to be a $\Phi$-factorization if in addition.

(C2.2.11c) The operator $f \mapsto (C^{-1}fM^{-1})M_-$ is bounded on $L^p$ on some augmented complete contour.

Again we call $\kappa_1, \ldots, \kappa_n$ the partial indices of $v$ and they are determined uniquely by $v$ up to the order. They are clearly independent of the augmentations of $\Gamma$ as long as $z_0$ is not on the augmented contours.

When $\Gamma$ is a complete contour, we may convert the form (C2.2.11b) to the form (C2.2.3b) by either setting $M(z) = D(z)m(z)$ for $z \in \Omega_+$, and $M(z) = m(z)$ for $z \in \Omega_-$ or setting $M(z) = (D(z))^{-1}m(z)$ for $z \in \Omega_+$ and $M(z) = m(z)$ for $z \in \Omega_-$.

We have the following extensions of Theorems 1.1.12, 1.1.19 (Q2.1).

Theorem 2.2.12. Assume that $v$ admits an (algebraic) factorization $v = v_+^{-1}v_-$. If $\mu$ is a solution of the following equation

$$\mu - h = C_{v\pm} \mu = C^+\mu(I - v_-) + C^-\mu(v_+ - I)$$

for some $h \in L^p(\Gamma)$ (or $h \in L^p(\Gamma) + L^\infty(\Gamma)$ in the case $v - I \in L^p$). then

$$H \equiv C_{v\pm} \mu(v_+^{-1} - v_-)$$

satisfies

(C2.2.15a) $H \in \mathcal{A}L^p(\mathcal{C} \setminus \Gamma)$.

(C2.2.15b) $H_+ + h = \nu v_\pm$ and hence $H_+ + h = (H_- + h)v$.

(C2.2.15c) $H \rightarrow 0$ as $z \rightarrow \infty$ along all curves nontangential to $\Gamma$.

Conversely, if a function $H$ satisfies (C2.2.15a) and (C2.2.15b), and if $H$ converges to 0 as $z \rightarrow \infty$ along any curve nontangential to $\Gamma$, then $\mu \equiv (H_+ + h)v_+^{-1}$ is a solution of Equation (2.2.13).

When $h = I$, $m \equiv H + I$ is a solution of the RH problem $(v, \Gamma)$ normalized at $z = \infty$ along any curves nontangential to $\Gamma$. We call $m \equiv H$ a vanishing solution of the RH problem $(v, \Gamma)$ in the case $h = 0$. The space of all vanishing solutions is isomorphic to $\ker(1 - C_{v\pm})$ by $H_\pm = \mu v_\pm$, $\mu \in \ker(1 - C_{v\pm})$. 
Theorem 2.2.16. Let $a \not\in \Gamma$ and $v = v_+^{-1}v_+$. If $\mu$ is a solution of the following equation

\[ (2.2.17) \quad \mu - h = C_{v_+,a}^{\text{def}} C_{\Gamma,a} \mu (I - v_-) + C_{\Gamma,a} \mu (v_+ - I) = 1 - C_{\Gamma,a}^{+} \mu v_- + C_{\Gamma,a}^{-} \mu v_+ \]

for some $h \in L^p(\Gamma)$, then

\[ (2.2.18) \quad H \equiv C_{\Gamma,a} \mu (v_+ - v_-) \]

satisfies

(C2.2.19a) $H \in L^p(\hat{\mathcal{C}} \setminus \Gamma)$,
(C2.2.19b) $H_{\pm} + h = \mu v_{\pm}$ and hence $H_{\pm} - h = (H_{\pm} - h) v_{\pm}$,
(C2.2.19c) $H(a) = 0$.

Conversely, if a function $H$ satisfies (C2.2.22abc), then $\mu \equiv (H_{\pm} + h) v_{\pm}^{-1}$ is a solution of Equation (2.2.17).

Again, when $h = I$, $m \equiv H + I$ is a solution of the RH problem $(v, \Gamma)$ normalized at $z = a$, and when $h = 0$, $m \equiv H$ is called a vanishing solution which represents an element of ker$(1 - C_{v_+})$.

The following theorem relates the singular integral operators constructed from two different factorizations of $v$:

\[ (2.2.20) \quad v = v_+^{-1}v_+ = \hat{v}_-^{-1} \hat{v}_+ \]

Theorem 2.2.21. Assume that $v$ admits two factorizations as in (2.2.20). Let $u = \hat{v}_-^{-1}v_+^{-1} = \hat{v}_-^{-1} \hat{v}_+^{-1} \in L^\infty$ and denote by

\[ U = \phi u \]

the multiplier from the right. Then we have the following relation

\[ (1 - C_{v_+,a}) U = (1 - C_{\hat{v}_+,a}) \phi \]

Proof.

\[ (1 - C_{v_+,a}) U \phi = \phi u - C_{\Gamma,a}^{+} \phi u (I - v_-) + C_{\Gamma,a}^{-} \phi u (I - v_+) \]
\[ = \phi u - C_{\Gamma,a}^{+} \phi (u - \hat{v}_+) - C_{\Gamma,a}^{-} \phi (\hat{v}_+ - u) \]
\[ = \phi - C_{\Gamma,a}^{+} \phi (I - \hat{v}_-) + C_{\Gamma,a}^{-} \phi (\hat{v}_- - I) = (1 - C_{v_+,a}) \phi \]

\[ \square \]

The next two theorems describe the relations of operators in connection to the augmentation of complete contours.

Theorem 2.2.22. Suppose that the orientation of a portion of $\Gamma$ is reversed to form a new contour $\Gamma'$ and $v$ is replaced by $v^{-1}$ there correspondingly to form a new jump matrix $v'$. Let $v = v_+^{-1}v_+$ be a factorization of $v$ and $v' = (v')^{-1}v'_+$ be the factorization so that $v'_\pm = v_\pm$ where the orientation has remained unchanged and $v'_\mp = v_\mp$ where the orientation has been reversed. Then

\[ (2.2.23) \quad C_{v_\pm,a,\Gamma} = C_{v'_\pm,a,\Gamma'} \]

Proof. (Q2.1) \[ \square \]
**Theorem 2.2.24.** Let $\Gamma$ and $\Gamma'$ be two contours with finite intersection and assume that $\nu_\gamma = 1$ on $\Gamma'$. Denote

$R_\Gamma$: the restriction $L^p(\Gamma \cap \Gamma') = L^p(\Gamma) \oplus L^p(\Gamma') \to L^p(\Gamma)$.

$\Gamma \to \Gamma_{\gamma}$: the embedding $L^p(\Gamma) \to L^p(\Gamma \cup \Gamma')$.

$C_R: L^p(\Gamma) \to L^p(\Gamma \cup \Gamma')$ the restriction of $C_{v_\gamma, a, \Gamma \cup \Gamma'}$ to $L^p(\Gamma)$.

We have

(2.2.25a) $C_{v_\gamma, a, \Gamma} R_\Gamma = R_\Gamma C_{v_\gamma, a, \Gamma \cup \Gamma'}$.

(2.2.25b) $C_{v_\gamma, a, \Gamma \cup \Gamma'} R_\Gamma = C_R C_{v_\gamma, a, \Gamma}$.

(2.2.25c) $\ker(1 - C_{v_\gamma, a, \Gamma \cup \Gamma'}) \cong \ker(1 - C_{v_\gamma, a, \Gamma})$ and $\coker(1 - C_{v_\gamma, a, \Gamma \cup \Gamma'}) \cong \coker(1 - C_{v_\gamma, a, \Gamma})$.

(2.2.25d) The operators $1 - C_{v_\gamma, a, \Gamma \cup \Gamma'}, 1 - C_{v_\gamma, a, \Gamma}$ are Fredholm if one of them is Fredholm, and they are invertible if one of them is invertible.

(2.2.25e) When the two operators in (2.2.25d) are invertible,

(2.2.26) $(1 - C_{v_\gamma, a, \Gamma})^{-1} = R_\Gamma (1 - C_{v_\gamma, a, \Gamma \cup \Gamma'})^{-1} R_\Gamma$.

(2.2.27) $(1 - C_{v_\gamma, a, \Gamma \cup \Gamma'})^{-1} = 1 + C_R (1 - C_{v_\gamma, a, \Gamma})^{-1} R_\Gamma$.

**Proof.** The proof for (2.2.25ab) are trivial. The first part of (2.2.25c) follows from the fact that both $\ker(1 - C_{v_\gamma, a, \Gamma \cup \Gamma'})$ and $\ker(1 - C_{v_\gamma, a, \Gamma})$ are representable by the same vanishing solutions of Theorem 2.2.16. While the second part follows from the facts that

$$\text{ran}(1 - C_{v_\gamma, a, \Gamma \cup \Gamma'}) = \text{ran}(1 - C_{v_\gamma, a, \Gamma}) \oplus L^p(\Gamma').$$

and

$$L^p(\Gamma \cup \Gamma') = L^p(\Gamma) \oplus L^p(\Gamma').$$

The results (2.2.25d) is simply a consequence of (2.2.25c).

To prove (2.2.25e), using (2.2.25ab), we obtain

$$R_\Gamma (1 - C_{v_\gamma, a, \Gamma \cup \Gamma'})^{-1} R_\Gamma (1 - C_{v_\gamma, a, \Gamma})^{-1} R_\Gamma R_\Gamma = (1 - C_{v_\gamma, a, \Gamma})^{-1} R_\Gamma.$$

and

$$1 + C_R (1 - C_{v_\gamma, a, \Gamma})^{-1} R_\Gamma = 1 + (1 - C_{v_\gamma, a, \Gamma \cup \Gamma'})^{-1} C_R R_\Gamma$$

$$= 1 + (1 - C_{v_\gamma, a, \Gamma \cup \Gamma'})^{-1} C_{v_\gamma, a, \Gamma \cup \Gamma'}$$

$$= (1 - C_{v_\gamma, a, \Gamma \cup \Gamma'})^{-1}.$$

As a corollary of Theorems 2.2.22, 2.2.24, we extend Theorem 2.2.5 to

**Theorem 2.2.8.** For any contour $\Gamma$, $v$ admits a $\Phi$-factorization if and only if the operator $1 - C_{v_\gamma, a}$ is Fredholm, and when this is the case, formulae (2.1-2.2) hold true.
Proposition 2.2.9. Let $\Gamma$ be any contour and consider the operators $1 - C_{v,a}$ and $1 - C_v$ as in $L^2 \otimes$ and $L^2 \otimes$ respectively. Then

\begin{equation}
(2.2.30a) \quad 1 - C_{v,a} = (z - a)(1 - C_v)(z - a)^{-1}.
\end{equation}

\begin{equation}
(2.2.30b) \quad \text{The two operators are Fredholm if one of them is Fredholm, and when they are Fredholm, their kernels and cokernels are isomorphic respectively.}
\end{equation}

Proof. The relation (2.2.30a) follows directly from (1.1.18c), and (2.2.30b) follows from (2.2.30a) and the fact that multiplier $(z - a)$ is clearly an isomorphism from $L^2 \to L^2$. \hfill $\Box$

A well known result in the GK theory is that if the problem $(v, R)$ with $v + v^* > 0$ admits a $\Phi$-factorization, then all the $L^2$ partial indices are zero. The following two theorems extend this result.

Theorem 2.2.31. Let $\Gamma$ be a reflection invariant contour: $\Gamma^* = \Gamma$. Assume that $v$ on $\Gamma$ admits a $\Phi$-factorization and satisfies

\begin{equation}
(2.2.32a) \quad v^*(z) = v(z) \quad \text{for} \quad z \in \Gamma \setminus R.
\end{equation}

\begin{equation}
(2.2.32b) \quad v^*(z) + v(z) \geq 0 \quad \text{for} \quad z \in R \quad \text{and} \quad v^*(z) + v(z) > 0 \quad \text{for} \quad z \text{ on a subset of } R \quad \text{with a positive measure.}
\end{equation}

Then $\Phi$ has all $L^2$ partial indices equal to zero.

Proof. Let $m$ be a vanishing solution of the $L^2$ RH problem $(v, \Gamma)$. Clearly, $H(z) \equiv m(z)m^*(z)$ is analytic off $\Gamma$, and has $L^1$ sense boundary values on $\Gamma$. We have therefore

\begin{equation}
0 = \int_{\Gamma} (H_+ + H_-) = \int_{\Gamma} (m_-v m^*_+ + m_-v^* m^*_+) = \int_{\Gamma} m_-(v + v^*) m^*_+.
\end{equation}

Thus by (2.2.32b) $m_-$ vanishes on a set of positive measure and therefore vanishes identically together with $m$. This shows that $\ker(1 - C_v) = 0$ in $L^2$. It then follows from Proposition 2.2.29 that $\ker(1 - C_{v,a}) = 0$ in $L^p$ and hence all partial indices are non-negative.

Now write the factorization $v = m_- Dm_+$ where we choose $z_+ = z_-$ and $z^{-}$ as the GK factorization $v^*(z) = m_+^*(z) z^* m_+^*(z)$ and $z^* m_-(z)^{-1}$. Clearly, $v^*(z)$ satisfies all conditions $v$ does and therefore all the partial indices are non-negative. \hfill $\Box$

The following theorem is useful for uniform controls of $\|(1 - C_v)^{-1}\|_{L^2}$ over certain parameters in the study of integrable systems.

Theorem 2.2.33. If in addition to the hypotheses of Theorem 2.2.31, $\Gamma = \mathbb{R}$ and $v + v^* > 0$, then

\begin{equation}
|| (1 - C_v)^{-1} ||_{L^2} \leq \frac{\lambda_{\max} + 1 + \sqrt{(\lambda_{\max} + 1)^2 - 4\lambda_{\min}}}{2\lambda_{\min}}.
\end{equation}

where

$\lambda_{\min} = \text{ess inf}_{z \in \mathbb{R}} \{ \text{minimal eigenvalue of } \frac{v(z) + v^*(z)}{2} \}$

and

$\lambda_{\max} = \text{ess sup}_{z \in \mathbb{R}} \{ \text{maximal eigenvalue of } \sqrt{v(z)v^*(z)} \}.$

Proof. Let $m$ and $h \in L^2$ be as in (2.11), then

$H(z) \equiv (m - h)(z)(m - h)^*(z)$
is analytic on $\mathbb{C} \setminus \mathbb{R}$ with boundary values in $L^1$ sense. Thus

$$0 = \int_{\mathbb{R}} H_+ = \int (m_+ m_-^* - m_+ h^* - h m_-^* + h h^*) = \int (m_- v m_-^* - m_- v h^* - h m_-^* + h h^*).$$

Adding this identity to its conjugate, we obtain

$$2\lambda_{\min} \|m_+\|^2 \leq \int m_- (v + v^*) m_-^* = \int (m_- v h^* + h v^* m_-^* + h m_-^* + m_- h^* - 2 h h^*) \leq 2(\lambda_{\max} + 1) \|m_-\| \|h\| - 2 \|h\|^2.$$

This proves that

$$\|m_+\|_{L^2} = \|(1 - C_v)^{-1} h\|_{L^2} \leq \frac{\lambda_{\max} + 1 + \sqrt{(\lambda_{\max} + 1)^2 - 4\lambda_{\min} \|h\|_{L^2}}}{2\lambda_{\min}}.$$

This estimates is particularly suitable for the study of integrable systems because the jump condition is often in the form of $v(z)$ conjugated by a unitary matrix $e^{z J - i J}$ and hence $\lambda_{\min}$ and $\lambda_{\max}$ depend only on $v(z)$.

Two jump matrices $v$ and $v'$ on a complete contour are said to be equivalent, $v \sim v'$, if $v = m_{-1}^* v' m_{-1}$ where $m_{-1}^* \in \mathbb{E}^\infty(\hat{\mathbb{C}} \setminus \Gamma)$. Clearly, if $v \sim v'$, then for any $1 < p < \infty$, they have the same $L^p$ partial indices. Furthermore, we have the following

**Proposition 2.2.35.** For any $1 < p < \infty$, the following operator relation holds

$$1 - C_{v_\pm} = (1 - C_{m_{\pm}})(1 - C_v),$$

where $v_+ = v' m_+$ and $v_- = m_-.$

The same relation holds true for $L^p$ space with $C$ in place of $C$ in (2.2.36).

**Proof.**

$$(1 - C_{m_{\pm}})(1 - C_v) = (1 - C_{m_{\pm}})(C^+ \hat{\phi} - C^- \hat{\phi} v') = C^+ (C^+ \hat{\phi} - C^- \hat{\phi} v') m_- = C^+ (\hat{\phi} + C^- \hat{\phi} - \hat{\phi} m_+ = C^+ \hat{\phi} - C^- \hat{\phi} v' m_+ = 1 - C_{v_\pm}.$$

### 2.3 Continuous $v$ for non-self-intersecting complete contours

We assume throughout this section that $\Gamma$ is a complete contour without self-intersections. For non-self-intersecting complete contours, the continuity of $v$ coincides with its classical sense. The smallest space with continuous $v$ studied in GK theory is the space of functions which are rational and bounded on each component of $\Gamma$. We denote this space by $R(\Gamma)$ and denote the space of piecewise rational matrix function on $\hat{\mathbb{C}} \setminus \Gamma$ by $R(\hat{\mathbb{C}} \setminus \Gamma).$ We knew from Theorems 2.1.7 and 2.2.5 that such a $v$ admits a $\Phi$-factorization. It is also easily seen that
Proposition 2.3.1. For \( v \in R(\Gamma) \), \( m \in R(\mathcal{C} \setminus \Gamma) \).

Of course, we also define rational GK factorization problems \((m, v_\gamma, z \in \mathcal{P})\) as for the RH problems in section 1.4.

By Theorems 2.1.7 and 2.2.5, every continuous \( v \) admits a \( \Phi \)-factorization in \( L^p \) space. In fact.

**Theorem 2.3.2.** Every continuous \( v \) admits an \( L^p \) \( \Phi \)-factorization and the partial indices are independent of \( 1 < p < \infty \).

**Proof.** Fix \( 1 < p < q < \infty \) By rational approximation, we write \( v = v^{(1)}v^{(2)} \) with \( \|v^{(1)} - I\|_{L^\infty} < \min(1/\|C^-\|_{L^q}, 1/\|C^-\|_{L^p}) \) and \( v^{(2)} \) rational. Thus \( v^{(1)} \) admits a factorization

\[
v^{(1)} = (m^{(-1)}_+)^{-1}m^{(1)}_+
\]

in both \( L^p \) and \( L^q \). Write \( v = b_-^{-1}b_+ \) where \( b_- = m^{(1)}_+ \) and \( b_+ = m^{(2)}v^{(2)} \). Thus the problem can be reduced to a rational problem \((M, b_\gamma, \mathcal{B}, z' \in \mathcal{P})\) which admits a factorization in the sense of Definition 2.2.10 where \( z_0 \) is chosen away from \( \Gamma \) and \( P_0 \). Since \( M \) is rational, \( m = M^\dagger Hb \) is a solution of the factorization problem in both \( L^p \) and \( L^q \) spaces in the sense of Definition 2.2.10. The partial indices are the same for \( p \) and \( q \) as they are determined by the same rational problem for \( M \).

**Corollary 2.3.3.** If \( v \) is continuous and satisfies conditions (2.2.32ab), then the RH problem \((v, \Gamma)\) admits an \( L^p \) solution for all \( 1 < p < \infty \).

The most beautiful part of the GK theory is the theory for decomposing subalgebras of continuous functions. The factors in \( L^p \) factorization stay in the same algebra. In the following we will briefly discuss the theory for the Sobolev space \( H^1 \), the theory for \( H^1 \), \( H^u \), \( H^o \), and other inverse closed decomposing algebras is similar. The basic results are

**Theorem 2.3.4.** Assume \( v \in H^1(\Gamma) \) (hence \( v^{-1} \) is \( H^1 \) by (C1.1.1)). Then

(2.3.5) The matrix function \( v \) admits a GK factorization

\[
v = m^{-1}Dm_+.
\]

where both \( m \) and \( m^{-1} \) have \( H^1 \) boundary values \( m_\pm \in \text{ran} C_\Gamma \).

(2.3.6) The operator \( 1 - C_\Gamma \) is Fredholm in \( H^1 \) and (2.2.6.7) hold true.

The proof of this theorem is similar to that of Theorem 2.3.2 and is left as an exercise (Q2.3).

2.4 Self-intersecting \( \Gamma \). In many important applications to inverse scattering theory, RH problems with self-intersecting \( \Gamma \) are involved. For \( L^\infty \) jump function \( v \), self-intersections do not make a difference in the theory. However, the classical concept of continuity of \( v \) is not appropriate for a very simple reason: no matter how smooth the solution \( m \) is up to the boundary, the jump function \( v = m^{-1}m_+ \) is in general discontinuous in the classical sense at self-intersections.

Let us first consider the rational problems with self-intersecting complete contours \( \Gamma \). We denote by \( R(\partial \Omega_\pm) \) the space of bounded matrix functions \( f \) on \( \Gamma \) such that

(C2.4.1) \( f \) is rational on the boundary of each component of \( \Omega_\pm \).

It is straightforward to verify (Q2.4) the following.
(2.4.2a) $C_{\pm}^\pm$ maps $R(\partial \Omega_{\pm}) \rightarrow R(\partial \Omega_{\pm})$.
(2.4.2b) $C_{\pm}^\pm$ maps $R(\partial \Omega_{\pm}) \rightarrow R(\partial \Omega_{\pm}) \cap R(\partial \Omega_{\mp})$.

Suppose that $v$ admits a GK factorization (1) with $m \in R(\tilde{\mathcal{C}} \setminus \Gamma)$. Clearly $v$ itself need not be a rational function. However, if we write $v = b^-_+b_+$ with $b_+ = m_-$ and $b_- = Dm_+$, we see that

$$v \in R(\partial \Omega_-) \cdot R(\partial \Omega_+)$$

This says that (2.4.3) is a necessary condition for $v$ to admit a GK factorization with $m \in R(\tilde{\mathcal{C}} \setminus \Gamma)$. Conversely, if (2.4.3) is true, we write $v = b^-_+b_+$ with $b_+ \in R(\partial \Omega_+)$ and reduce as in Theorem 1.4.4 the problem into a problem with a non-self-intersecting complete contour. Thus we have the following theorem.

**Theorem 2.4.4.** The jump matrix $v$ admits a GK factorization with $m \in R(\tilde{\mathcal{C}} \setminus \Gamma)$ if and only if $v \in R(\partial \Omega_-) \cdot R(\partial \Omega_+)$.

**Remark 2.4.4a.** $R(\partial \Omega_-) \cdot R(\partial \Omega_+)$ in general is not an algebra while $R(\partial \Omega_-)$ and $R(\partial \Omega_+) \cap R(\partial \Omega_-)$ are.

For the same reason, it is not appropriate to study the GK factorization theory by using a single space of continuous functions when the complete contour $\Gamma$ has self-intersections. We use a pair of spaces $C(\Omega_\pm)$ consisting functions $f$ which satisfy

(2.4.4b) $f$ is continuous on the boundary of each component of $\Omega_\pm$.

**Theorem 2.4.4c.** Assume that $v$ admits an (algebraic) factorization $v = v_1^{-1}v_2$. $v_\pm \in C(\Omega_\pm)$, then for any $1 < p < \infty$,

(2.4.4d) The function $v$ admits a GK factorization with $m_\pm \in L^p_1(\Gamma)$.

(2.4.4e) The operator $1 - C_{\pm}$ applied on $r \times n$ matrix functions in $L^p(\Gamma)$ is Fredholm and

$$\dim \ker(1 - C_{\pm}) = -1 \sum_{\kappa_1 > 0} \kappa_j, \quad \dim \ker(1 - C_{\pm}) = r \sum_{\kappa_1 > 0} \kappa_j,$$

and therefore

(2.4.4f) $\kappa(1 - C_{\pm}) = \kappa(v) = r(W_1 \det v_+ - W_1 \det v_-) = -r(\kappa_1 + \cdots + \kappa_n)$.

**Proof.** We compute

(2.4.4g) $$(1 - C_{\pm}^\pm)(1 - C_{\pm}^\pm) = (1 - C_{\pm}^\pm)(C^- \circ v_+ - C^- \circ v_-)$$
$$= C^- + (C^- \circ v_+ - C^- \circ v_-) + C^- + (C^- \circ v_+ - C^- \circ v_-) + 1 \equiv 1 - T_v.$$

The compactness of the operator $T_v$ on $L^p$ follows from the uniform approximation of $v_\pm^{-1}$ by functions in $R(\partial \Omega_{\pm})$. Thus $1 - C_{\pm}^\pm$ is a left regulator for $1 - C_{\pm}$. Interchanging $v$ and $v^{-1}$ in (2.4.11) shows that $1 - C_{\pm}^\pm$ is a right regulator as well.
This prove the Fredholmness of the operator $1 - C_{v}$; the rest of the theorem is standard in GK factorization theory.

It is evident that to extend the GK factorization theory for decomposing algebras to include complete contours $\Gamma$ with self-intersections, working with a single decomposing algebra is no longer suitable. We will instead work with a pair of decomposing algebras which include $R(\partial \Omega_{-})$ and $R(\partial \Omega_{+})$ respectively.

We will take the $L^{2}$ Sobolev space $H^{k}$ as an example while the theory for other decomposing algebras is similar. We define the Sobolev space $H^{k}$ as follows. Let $\omega$ be an open connected region in $\mathbb{C}$ with piecewise smooth boundary $\partial \omega$ without self-intersections. Denote by $S_{\omega}$ the set of nonsmooth points on $\partial \omega$. The space $H^{k}(\partial \omega)$, $k \geq 0$ consists of matrix functions $f$ on $\partial \omega$ which satisfies

(C2.4.5a) The distributional derivatives $f^{(j)} \in L^{2}$ for $0 \leq j \leq k$ on each curve segment of $\partial \omega \setminus S_{\omega}$.

(C2.4.5b) At each point $z' \in S_{\omega}$, $f^{(j)}$, $0 \leq j \leq k - 1$ matches from the two sides. (This makes sense because these functions are H"older continuous on each curve segment by (C2.4.5a)).

Now we define $H^{k}(\partial \Omega_{\pm})$ as the space of functions $f$ which satisfies

(C2.4.6a) For each component $\omega$ of $\Omega_{\pm}$, $f \in H^{k}(\partial \omega)$.

$H^{k}(\partial \Omega_{\pm})$ is a Hilbert space with the inner product

$$ (f, g) = \int \sum_{j=0}^{k} \operatorname{tr} f^{(j)}(g^{(j)})^{*} |dz|. $$

Similar to (2.4.2ab), we have (Q2.4.7)

(2.4.7a) $C^{\pm}$ is bounded from $H^{k}(\partial \Omega_{\pm}) \to H^{k}(\partial \Omega_{\pm})$.

(2.4.7b) $C^{\pm}$ is bounded from $H^{k}(\partial \Omega_{\pm}) \to H^{k}(\partial \Omega_{\pm}) \cap H^{k}(\partial \Omega_{\mp})$.

Gain, if $\infty \in \Gamma$, we extend these spaces to

$$ H^{k}_{\Gamma}(\partial \Omega_{\pm}) \overset{\text{def}}{=} M(n, \mathbb{C}) \oplus H^{k}(\partial \Omega_{\pm}), $$

and extend $C^{\pm}_{\Gamma}$ such that for constant functions $f$, $C^{+}\!\! f = f, C^{-} f = 0$. For consistency, we also write $H^{k}_{\Gamma}(\partial \Omega_{\pm}) = H^{k}(\partial \Omega_{\pm})$ when $\infty \notin \Gamma$.

**Theorem 2.4.9.** Assume that $v$ admits an (algebraic) factorization $v = v_{-}^{-1} v_{+}$.

$v_{-} \in H^{k}_{\Gamma}(\partial \Omega_{-})$, then

(2.4.10a) The function $v$ admits a GK factorization with $m_{\pm} \in H^{k}_{\Gamma}(\partial \Omega_{\pm})$.

(2.4.10b) The operator $1 - C_{v}$ is Fredholm on $H^{k}(\partial \Omega_{+}) \cap H^{k}(\partial \Omega_{-})$.

(2.4.10c) The operator $1 - C_{v_{-}}$, applied on $\tau \times n$ matrix functions in $H^{k}(\partial \Omega_{+}) \cap H^{k}(\partial \Omega_{-})$, is Fredholm and

$$ \dim \ker(1 - C_{v}) = -\tau \sum_{\kappa_{j} < 0} \kappa_{j}, \quad \dim \coker(1 - C_{v}) = \tau \sum_{\kappa_{j} > 0} \kappa_{j}, $$

and therefore

$$ \kappa(1 - C_{v}) = -\tau \kappa(v) = -\tau(W_{\Gamma} \det v_{+} - W_{\Gamma} \det v_{-}) = -\tau(\kappa_{1} + \cdots + \kappa_{n}). $$

**Proof.** Using (2.4.7ab), we see that $1 - C_{v_{+}}, 1 - C_{v_{-}} \in H^{k}(\partial \Omega_{+}) \cap H^{k}(\partial \Omega_{-})$. By (2.4.4h),

$$ (1 - C_{v_{+}})(1 - C_{v_{-}}) = 1 - T_{v}. $$
The compactness of \( T_r \) can be seen from the approximation of \( v_\pm^{-1} \) by functions in \( R(\partial \Omega_\pm) \). Thus \( 1 - C_{v_\pm} \) is a left regulator for \( 1 - C_{v_\pm} \). Interchanging \( v \) and \( v^{-1} \) in (2.4.11) shows that \( 1 - C_{v^{-1}} \) is a right regulator as well. This proves (2.4.10b).

By rational approximation we write

\[ v_\pm = v^{(1)}_\pm v^{(2)}_\pm, \]

where \( v^{(2)}_\pm \in R(\partial \Omega_\pm) \), \( v^{(1)}_\pm - I \in H^k(\partial \Omega_+) \cap H^k(\partial \Omega_-) \), and \( v^{(1)}_\pm - I \) is sufficiently small in \( L^\infty \) norm so that \( v^{(1)}_\pm(z,t) = (I + t(v^{(1)}_\pm(z,t) - I)) \) is invertible for all \( z \in \Gamma \) and \( 0 \leq t \leq 1 \) and that the \( L^2 \) partial indices of \( v^{(1)}_\pm \) are all equal to zero. Hence \( v^{(1)}_\pm \) admits an \( L^2 \) GK factorization \( v^{(1)}_\pm = (m^{(1)}_\pm)^{-1} m^{(1)}_\pm \), where \( m^{(1)}_\pm \) is normalized at \( \infty \). Also ker\((1 - C_{v^{(1)}_\pm}) = 0 \) in \( L^2 \) and hence in \( H^k(\partial \Omega_+) \cap H^k(\partial \Omega_-) \). Since \( v^{(1)}_\pm \) deforms continuously into \( I \) through \( v^{(1)}_\pm(t) \), the Fredholm index of \( 1 - C_{v^{(1)}_\pm} \) is zero in \( H^k(\partial \Omega_+) \cap H^k(\partial \Omega_-) \). Thus \( 1 - C_{v^{(1)}_\pm} \) is invertible in \( H^k(\partial \Omega_+) \cap H^k(\partial \Omega_-) \) and \( m^{(1)}_\pm \) can be obtained through \( m^{(1)}_\pm = \mu v^{(2)}_\pm \in I + H^k(\partial \Omega_+) \) where

\[ \mu = (1 - C_{v^{(1)}_\pm})^{-1} I \in I + H^k(\partial \Omega_+) \cap H^k(\partial \Omega_-). \]

Setting \( b = m^{(1)}_\pm v^{(2)}_\pm \), we have \( v = b^{-1} b \). Now as in Theorem 1.4.4, the problem reduces to the rational problem \( (H, b, z \in R_b) \) which admits a factorization in the sense of Definition 2.2.10. Hence \( m \equiv Hb \) gives a GK factorization in the form of Definition 2.2.10.

Finally, since \( m \) and \( m^{-1} \) are bounded. \( v \sim D \) and (C2.4.20de) folow from Proposition 2.2.35. □

We remark that in this proof only \( L^\infty \) approximation is used and hence it also works for the \( \infty \) space which rational functions are not dense.

The space \( H^k(\partial \Omega_\pm) = H^k(\partial \Omega_\pm, 2|dz|/(1 + |z|^2)) \) is defined analogously with \( |dz| \) replaced by \( 2|dz|/(1 + |z|^2) \) in (2.4.6b).

**Definition 2.4.12.** An \( n \times n \) matrix jump function \( v \) on \( \Gamma \) is said to be continuous at \( c \in \Gamma \) if

(2.4.13a) \( v(z) \) and \( v^{-1}(z) \) both have limits \( v_{\nu,c} \) and \( v_{\mu,c}^{-1} \) respectively along each ray \( \gamma_{\nu}, \nu = 0, \ldots, k \equiv 0 \) \( \mod k \), ordered counterclockwise, joining at \( c \).

(2.4.13b) \( v_{\nu,c} = \cdots v_{1,c} \cdots v_{k-1,c} = I \), where for \( j = 0, \ldots, k \),

\[ \begin{align*}
\nu_{\nu,c} & \quad \text{if } \gamma_{\nu} \text{ is oriented outwards} \\
\nu_{\nu,c}^{-1} & \quad \text{if } \gamma_{\mu} \text{ is oriented inwards}
\end{align*} \]

Clearly, if \( c \in \Gamma \) is not at a self-intersection and not an end point, the above definition simply says that both \( v \) and \( v^{-1} \) are continuous in classical sense.

A jump function \( v \) is said to be continuous on \( \Gamma \) if it is continuous everywhere on \( \Gamma \).

Similarly, we have

**Definition 2.4.14.** An \( n \times n \) matrix jump function \( v \) on \( \Gamma \) is said to be \( k \)-smooth at \( c \in \Gamma \) if
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(2.4.14a) \( v(z) \) and \( v^{-1}(z) \) both have Taylor expansions of order \( k \), \( v_{\nu,c} \) and \( v_{\nu,c}^{-1} \) respectively along each ray \( \gamma_{\nu}, \nu = 0, 1, \ldots, k \equiv 0 \mod k \), ordered clockwise, joining at \( c \);

(2.4.14b) \( \hat{v}_{\nu,c} \hat{v}_{1,c} \cdots \hat{v}_{k-1,c} = I + O((z-c)^{k+1}) \) (or \( O(z^{-k-1}) \) if \( c = \infty \)), where for \( j = 0, \ldots, k \),

\[
\hat{v}_{\nu,c} = \begin{cases} 
v_{\nu,c} & \text{if } \gamma_{\nu} \text{ is oriented outwards} \\
v_{\nu,c}^{-1} & \text{if } \gamma_{\nu} \text{ is oriented inwards.}
\end{cases}
\]

Note that the \( k \)-smoothness is not a smoothness in classical sense. It does not imply the existence of derivatives. We use it only in connection with the following proposition, which gives a criterion on functions in \( H_{+}^{k}(\partial \Omega_{-}) \cdot H_{+}^{k}(\partial \Omega_{+}) \).

**Proposition 2.4.15.** A jump function \( \nu \) on a complete contour \( \Gamma \) belongs to the space \( H_{+}^{k}(\partial \Omega_{-}) \cdot H_{+}^{k}(\partial \Omega_{+}) \) if and only if \( \nu \) is \( H_{+}^{k} \) away from self-intersections and is \( k \)-smooth at each finite self-intersection of \( \Gamma \).

**Proposition 2.4.16.** If \( v \) is continuous, then after augmenting \( \Gamma \) to a complete contour \( \tilde{\Gamma} \) as in (Q.1.), the extended \( v \) remains continuous.

The GK factorization theory for continuous \( v \) on non-self-intersecting complete contours can be extended to the case of any contour \( \Gamma \) based on the following observations.

**Proposition 2.4.17.** Let \( \Gamma \) be a complete contour and \( v \) be continuous, then \( v \) admits the (algebraic) factorization

\[
v = v_{-}^{-1}v_{+},
\]

where \( v_{\pm} \) is continuous on each component of \( \partial \Omega_{\pm} \).

If \( v \) is continuous, then the \( RH \) problem \( (v, \Gamma) \) admits an \( L^{p} \) GK factorization for any \( 1 < p < \infty \) and the partial indices are independent of \( p \).

2.4 Piecewise Continuous Problems.

**Exercises.**

(Q2.1) Let \( v = z^{k} \) and \( \Gamma = S^{1} = \{ |z| = 1 \} \). Compute the kernel and the cokernel of \( 1 - C_{v} \) explicitly in the \( L^{2} \) space.

Hint: Use the Fourier expansion \( f(z) = \sum_{-\infty}^{\infty} a_{n} z^{n} \) for \( f \in L^{2}(S^{1}) \).

(Q2.2) Let \( v \) be continuous on a simple closed complete contour \( \Gamma \). Show that \( v \) can be continuously deformed to \( \text{diag}[1, \ldots, 1, \det v] \).

Hint: Deform \( v \) first through a small perturbation to \( v' \) so that \( v' \) is rational with \( v' \neq 0 \).

(Q2.2) Assume (2.2.3abc). Show that the operator

\[
(C^{+} \circ m_{-}^{-1})m_{-} + C^{-} \circ m_{-}^{-1})m_{-} \nu
\]

is a two sided regulator for \( 1 - C_{v} \).

(Q2.2) Verify formulae (2.2.6, 2.2.7) directly for \( v = D(z) \).

(Q2.1) Show that the rational problem for \( (m, v \in R(\Gamma)) \) on a non-self-intersecting complete contour \( \Gamma \) can be reduced (deformed) to a rational problem of the form \( (H, u_{x} \in V_{x}, x' \in P) \), where \( H \) is the analytic continuation of \( m \) on a component of \( \mathcal{C} \setminus \Gamma \).
3.1. Forward Scattering. The AKNS–ZS system is a $2 \times 2$ system of the form

\begin{equation}
\psi_x = iz\sigma\psi + q(x)\psi.
\end{equation}

where

\[ \sigma = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad q(x) = \begin{pmatrix} 0 & Q(x) \\ -Q(x) & 0 \end{pmatrix}. \]

The scattering problem is to find, for a given potential $q$, certain piecewise analytic matrix solutions $\psi$ and the multiplicative jump matrix $\nu$ which will be called the scattering data. The map $q \mapsto \nu$ is called the scattering transform of of system 3.1.1. In this section, we construct the scattering transform from the space $L^2((1+x^2)dx)$ to $H^1(dx)$. When $q = 0$, the fundamental solution $\psi = e^{izx}$ has an essential singularity at $z = \infty$. We therefore make a change of variable

\begin{equation}
\psi = me^{izx}.
\end{equation}

and accordingly system 3.1.1 changes to

\begin{equation}
m_x = iz\text{ad}\sigma m + qm.
\end{equation}

It is useful to remember that if $m$ is a fundamental solution and $m^{(1)}$ a solution, then

\begin{equation}
m^{-1}(x, z)m^{(1)}(x, z) = e^{izx \text{ad}\sigma} A(z),
\end{equation}

for some matrix function $A(z)$. This system can be further rewritten as

\[(e^{-izx \text{ad}\sigma} m)_x = e^{-izx \text{ad}\sigma} qm\]

or in the form of the integral equation

\begin{equation}
m(x, z) = e^{izx \text{ad}\sigma} A(z) + \int_{x_0}^x e^{i(z-y)x \text{ad}\sigma} q(y)m(y, z)dy,
\end{equation}

where $A(z)$ is any matrix function independent of $x$ and the lower limit $x_0$ can be different for different entries of the matrix function. We will use a number of solutions of (3.1.5) in order to study the scattering transform. The following diagram indicates the relations of these solutions and their functions in the scattering theory.
Consider the following two Volterra integral equations as special cases of (3.1.5) for real $z$.

\begin{equation}
(3.1.6) \quad m^{(\pm)}(x, z) = I + \int_{-\infty}^{x} e^{i(z-y)z \sigma} q(y) m^{(\pm)}(y, z) dy \equiv I + K_{q, z, \pm} m^{(\pm)}.
\end{equation}

Clearly, these equations have continuous bounded solutions for real $z$ when $q \in L^1$.

For nonreal $z$, observing that for any $2 \times 2$ matrix $(a_{ij})$,

\begin{equation}
(3.1.7) \quad e^{i(z-y)z \sigma} (a_{ij}) = \begin{pmatrix}
0 & a_{12} \\
-a_{21} & 0
\end{pmatrix}
\end{equation}

we see that $e^{i(z-y)z \sigma}$ is bounded for $\text{Im} \, z \leq 0$ ($\geq 0$) when applied on the second (first) column of (3.1.6a) and the first (second) column of (3.1.6b). Thus writing

\begin{equation}
(3.1.8) \quad m^{(\pm)} = m^{(\pm)}_1 + m^{(\pm)}_2, \quad m^{(\pm)}_1 \overset{\text{def}}{=} m^{(\pm)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad m^{(\pm)}_2 \overset{\text{def}}{=} m^{(\pm)} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\end{equation}

we have, for $q \in L^1$. 
(3.1.9) The equations for $m_1^{(±)}$ and $m_2^{(±)}$ are analytic for $± \Im z > 0$.

Let $M$ be a measure space and $B$ a Banach (Fréchet) space. We denote by $B \otimes L^p(M)^{\infty} \otimes L^p(M) \rightarrow B$ the space of $B$-valued $L^p$ functions with norm (seminorms)
\[ \|f\|_{B \otimes L^p(M)} = \|f\|_{B \otimes L^p(M)}^{\infty} = \|f\|_{L^p(B; L^p(M))}. \]
We often use the following results:

(3.1.9a) (Minkowski) $\|f\|_B \leq \|f\|_{B \otimes L^1(M)}$.
(3.1.9b) (Q3.1.9) $\|f\|_{L^p(M)} \leq \|f\|_{L^p(B; L^p(M))}$, $p \leq q$.
(3.1.9c) (Fubini) $\|f\|_{L^p(B; L^p(M))} \leq \|f\|_{L^p(B; L^p(M))}$.

The following theorem gives certain estimates needed for the inverse problem.

**Theorem 3.1.10.** Assume that

(3.1.11) $q \in L^2((1 + x^2)dx)$

and write

(3.1.12) $M(x, z) = \begin{cases} m_1^{(±)} + m_2^{(±)}, & \text{Im } z > 0, \\ m_1^{(±)} + m_2^{(±)}, & \text{Im } z < 0. \end{cases}$

We have

(3.1.13a) $\varphi(x, y) \leq e^{-c|x-y|} |z| \varepsilon$, where $c$ is some positive constant.
(3.1.13b) $M_\pm - I \in (L^2_\pm, L^2_\pm) \otimes C^\infty L^2(\mathbb{R}, dz)$, where $L^2_\pm$ denotes the space of functions in $L^2((a, \infty))$ ($L^2((-\infty, a))$) for all $a \in \mathbb{R}$ and $L^2_\pm$ denotes the space of $2 \times 2$ matrix functions whose first column is in $L^2_\pm$ and the second column in $L^2_\pm$.

(3.1.13c) For each $x \in \mathbb{R}$, $M_\pm(x, \cdot) - I \in C^\infty H^1(\mathbb{R}, dz)$

**Lemma 3.1.14.** Hardy's inequality. See also (Q3.1.14). Let $h(x) = \int_{-\infty}^{\infty} f(y)dy$ with $f \in L^1((a, \infty))$, then

(3.1.15) $\|h\|_{L^2((-\infty, a))} \leq 2\|(a-\cdot)h\|_{L^2((-\infty, a))}$.

The same result is true when the $-\infty$ is replaced by $\infty$.

**Proof.** Without loss we assume that $f$ is a real valued scalar function. Then

\[
\int_{-\infty}^{a} h^2 dx = \int_{-\infty}^{a} dx \int_{-\infty}^{x} 2h(y)h'(y)dy \\
= 2 \int_{-\infty}^{a} dx \int_{-\infty}^{x} h(y)f(y)dy \\
= 2 \int_{-\infty}^{a} (a-x)h(x)f(x)dx \\
\leq 2\|h\|_{L^2((-\infty, a))}\|(a-\cdot)f\|_{L^2((-\infty, a))}.
\]

**Proof of Theorem 3.1.10.** Statements (3.1.13ab) are standard results of Volterra integral equations which can be obtained directly from the iterations.
We will only prove (3.1.13cd) for \( m_1^{(+)} \). From (3.1.6a),
\[
(3.1.16) \quad m_1^{(+)}(x, z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \int_{+\infty}^{z} e^{i(x-y)z} \text{ad} \sigma q(y) m_1^{(-)}(y, z)dy \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + K_{q, z, +} m_1^{(+)}.
\]
By the \( L^2 \) theory of the Fourier transform,
\[
K_{q, z, +} \left( \begin{array}{c} 1 \\ 0 \\
0 \\ 0 \end{array} \right) = \int_{+\infty}^{z} \left( \begin{array}{c} 0 \\ -e^{-i(x-y)z} Q(y) \\ 0 \\ 0 \end{array} \right) dy
\]
belongs to \( C^+ L^2(dz) \) for each fixed \( x \) and
\[
\|K_{q, z, +} \left( \begin{array}{c} 1 \\ 0 \\
0 \\ 0 \end{array} \right) \|_{L^2(dz)} = \left( \int_{+\infty}^{x} |Q|^2 dy \right)^{\frac{1}{2}}
\]
We see that
\[
(3.1.17) \quad K_{q, z, +} \left( \begin{array}{c} 1 \\ 0 \\
0 \\ 0 \end{array} \right) \in C^+L^2(dz) \otimes (C(\mathbb{R}) \cap L^\infty(dz)) \cap (C(\mathbb{R}) \cap L^\infty(dz)).
\]
By standard iteration method for Volterra integral equations, one can show that the operator \( (1 - K_{q, z, +})^{-1} \) is bounded, uniformly for \( \text{Im } z \geq 0 \), in the space of continuous functions and analytic in \( z \) in the upper half complex plane. Thus
\[
(3.1.18) \quad m_1^{(-)} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (1 - K_{q, z, +})^{-1} K_{q, z, +} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in C^+ L^2(dz) \otimes (C(\mathbb{R}) \cap L^\infty)
\]
Using (3.1.6) and Lemma 3.1.14, we see that in fact
\[
(3.1.19) \quad \text{LHS(3.1.18)} \in C^+ L^2(dz) \otimes (C(\mathbb{R}) \cap L^\infty \cap L^2_h(dz)).
\]
Since \( \partial_z m_1^{(+)} \) satisfies the equation
\[
\partial_z m_1^{(+)}(x, z) = \int_{+\infty}^{z} i(x-y) \text{ad} \sigma e^{i(x-y)z} \text{ad} \sigma q(y) m_1^{(+)}(y, z)dy + \int_{+\infty}^{z} e^{i(x-y)z} \text{ad} \sigma q(y) \partial_z m_1^{(+)}(y, z)dy,
\]
the function
\[
\hat{m} \overset{\text{def}}{=} (\partial_z - ix \text{ad } \sigma) m_1^{(+)}
\]
satisfies the equation
\[
\hat{m}(x, z) = -i \int_{+\infty}^{z} ye^{i(x-y)z} \text{ad} \sigma (\text{ad } \sigma q(y)) m_1^{(+)}(y, z)dy + \int_{+\infty}^{z} e^{i(x-y)z} \text{ad} \sigma q(y) \hat{m}(y, z)dy
\]
\[
\equiv h_1 + h_2 + K_{q, z, +} \hat{m}.
\]
where

$$h_1 = -i \int_{+\infty}^{(x)} ye^{i(x-y)\sigma} (ad \sigma q(y)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} dy$$

$$= \int_{+\infty}^{(x)} \begin{pmatrix} 0 \\ ie^{i(x-y)\sigma} yQ(y) \end{pmatrix} dy$$

$$\in C^+L^2(dz) \otimes C(\mathbb{R}).$$

and

$$h_2 = -i \int_{+\infty}^{(x)} ye^{i(x-y)\sigma} (ad \sigma q(y))(m_1^{(+)}(y, \langle z \rangle) - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) dy \in C^+L^2(dz) \otimes C(\mathbb{R}).$$

by (3.1.19). Hence $h \in C^+L^2(dz) \otimes C(\mathbb{R})$ and

$$\hat{m} = (1 - K_{q,z,\sigma})^{-1} h \in C^+L^2(dz) \otimes C(\mathbb{R}).$$

Now (3.1.13d) follows from the fact that $\|m(x, \cdot)\|_{L^2} + \|\hat{m}(x, \cdot)\|_{L^2}$ is equivalent to the $H^1$ norm of $m(x, \cdot)$.

If we take $M$ as the piecewise analytic matrix function for the RH problem, the jump matrix (scattering data) should be supported at the following three types of singularities.

(3.1.20a) A discrete set of nonreal $z$ on which $\det M$ vanishes. These $z$'s are the $L^2$-eigenvalues of (3.1.1) (Q3.1.20a) which give rise to solitons for related integrable systems. There can be a finite or an infinite number of eigenvalues for system (3.1.1).

(3.1.20b) A set of $z = 0$ on which $\det M_\pm$ vanishes. These $z$'s are called the spectral singularities. There can be a finite or an infinite number of spectral singularities for system (3.1.1).

(3.1.20c) The jump $M_\pm^* M_\pm$ on $\mathbb{R}$.

A more detailed knowledge of these singularities is needed in order to set up the RH problem. However, it is more convenient to choose certain matrix functions $m$ which are normalized as either $x \to -\infty$ or $x \to +\infty$ (hence $\det m \equiv 1$ by the theory of ODE). We proceed by defining a matrix function $A(z)$ for real $z$

$$\psi^{(+)} = \psi^{(-)} A(z), \quad A(z) = \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix},$$

where $\psi^{(\pm)} = m^{(\pm)} e^{iz \sigma}.$

**Theorem 3.1.22.**

(3.1.23a) $A - I \in H^1(\mathbb{R})$ and $\det A(z) \equiv 1$.

$$\begin{cases}
a = \det(\psi_1^{(+)}, \psi_2^{(-)}) = \det M, & \text{Im} z \geq 0,
\bar{a} = \det(\psi_1^{(-)}, \psi_2^{(+)}) = \det M, & \text{Im} z \leq 0,
\end{cases}$$

(3.1.23b)

$$\begin{cases}
b = \det(\psi_1^{(-)}, \psi_1^{(+)}), & \text{Im} z = 0,
\bar{b} = \det(\psi_2^{(+)}, \psi_2^{(-)}), & \text{Im} z = 0.
\end{cases}$$
where \( \psi^{(\pm)} = (\psi_1^{(\pm)}, \psi_2^{(\pm)}) \).

(3.1.23c) \( a - 1 \in C^+ H^1, \ a - 1 \in C^- H^1 \).

\[
\begin{aligned}
\begin{cases}
    a(z) = 1 - \int_R Q(y)m_{11}^{(+)}(y, z) dy = 1 - \int Qm_{12}^{(-)}, & \text{Im } z \geq 0, \\
    \hat{a}(z) = 1 + \int_R \hat{Q}(y)m_{11}^{(+)}(y, z) dy = 1 + \int Qm_{21}^{(-)}, & \text{Im } z \leq 0, \\
    b(z) = \int_R \hat{Q}(y)e^{iyz}m_{11}^{(+)}(y, z) dy = \int Qe^{iyz}m_{11}^{(-)}, & \text{Im } z = 0, \\
    \hat{b}(z) = -\int_R Q(y)e^{-iyz}m_{12}^{(+))(y, z) dy = -\int Qe^{-iyz}m_{22}^{(-)}, & \text{Im } z = 0.
\end{cases}
\end{aligned}
\]

where \( m^{(\pm)} = \begin{pmatrix} m_{11}^{(\pm)} & m_{12}^{(\pm)} \\ m_{21}^{(\pm)} & m_{22}^{(\pm)} \end{pmatrix} \).

(3.1.23e) For \( \text{Im } z > 0 \),

\[
M \rightarrow \begin{cases}
    \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, & x \rightarrow -\infty, \\
    \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, & x \rightarrow +\infty.
\end{cases}
\]

For \( \text{Im } z < 0 \),

\[
M \rightarrow \begin{cases}
    \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, & x \rightarrow -\infty, \\
    \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, & x \rightarrow +\infty.
\end{cases}
\]

(3.1.23f) For real \( z \),

\[
M_- \sim \begin{cases}
    \begin{pmatrix} a & 0 \\ e^{-iz}b & 1 \end{pmatrix}, & x \rightarrow -\infty, \\
    \begin{pmatrix} 1 & -e^{iz}b \\ 0 & a \end{pmatrix}, & x \rightarrow +\infty.
\end{cases}
\]

\[
M_- \sim \begin{cases}
    \begin{pmatrix} 1 & e^{iz}b \\ 0 & a \end{pmatrix}, & x \rightarrow -\infty, \\
    \begin{pmatrix} a & 0 \\ -e^{-iz}b & 1 \end{pmatrix}, & x \rightarrow +\infty.
\end{cases}
\]

**Proof.** Since \( \text{tr}(iz\sigma + q(x)) = 0 \),

\( \text{det } \psi \) for system (3.1.11) is constant in \( z \). It follows then from the fact that \( \lim_{z \rightarrow \pm \infty} m^{(\pm)} = I \), that \( \text{det } \psi^{(\pm)} = 1 \) and therefore \( \text{det } A(z) = 1 \). Using this fact and (3.1.13d), we obtain \( A \in H^1 \). The formulae in (3.1.23b) deduce directly from (3.1.21) using (3.1.23a). Clearly (3.1.23c) follows from (3.1.23b) and (3.1.23d) is obtained by taking the limits \( x \rightarrow \pm \infty \) on (3.1.21b).

For (3.1.23e) in the case \( \text{Im } z > 0 \), \( M = (m_1^{(+)}, m_2^{(-)}) \). As \( x \rightarrow -\infty \), \( m_1^{(-)} \rightarrow 0 \), \( m_1^{(+) \rightarrow a} \) by (3.1.23d), and

\[
\int_x^{+\infty} \hat{Q}(y)e^{-iz(y-x)}m_{11}^{(+)(y, z) dy \rightarrow 0}
\]
by dominated convergence theorem. We have proved that

\[ M \rightarrow \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad x \rightarrow -\infty, \quad \text{Im} \ z > 0. \]

The proof for the rest of (3.1.23e) is similar. Finally, (3.1.23f) follows directly from (3.1.6) and (3.1.23d). \( \square \)

We can now define a solution \( m \) normalized as \( x \rightarrow -\infty \) and a solution \( \tilde{m} \) normalized as \( x \rightarrow +\infty \) for \( \text{Im} \ z > 0 \) because of (3.1.23e):

\[ \begin{aligned}
   M &= \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{Im} \ z > 0, \\
   M &= \begin{pmatrix} 1 & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix}, \quad \text{Im} \ z < 0.
\end{aligned} \tag{3.1.24} \]

\[ \begin{aligned}
   \tilde{m} &= \begin{pmatrix} 1 & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix}, \quad \text{Im} \ z > 0, \\
   \tilde{m} &= \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{Im} \ z < 0.
\end{aligned} \tag{3.1.25} \]

Define \( v_\pm \) and \( \tilde{v}_\pm \) (see (3.1.4)) by:

\[ e^{i z z \cdot a d} v_\pm(z) = (m^{(z)})^{-1} m_\pm. \tag{3.1.26} \]

\[ e^{i z z \cdot a \bar{a}} \tilde{v}_\pm(z) = (\tilde{m}^{(z)})^{-1} \tilde{m}_\pm. \tag{3.1.27} \]

Taking the limits \( x \rightarrow \pm \infty \), we compute using (3.1.23f).

\[ v_+ = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b/a & 1 \end{pmatrix}. \tag{3.1.28} \]

\[ v_- = \begin{pmatrix} 1 & \bar{b} \\ 0 & \bar{a} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \bar{a}^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \bar{b}/a \\ 0 & 1 \end{pmatrix}. \tag{3.1.29} \]

\[ \tilde{v}_+ = \begin{pmatrix} 1 & -\bar{b} \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & -\bar{b}/a \\ 0 & 1 \end{pmatrix}. \tag{3.1.30} \]

\[ \tilde{v}_- = \begin{pmatrix} \bar{a} & 0 \\ -\bar{b} & 1 \end{pmatrix} \begin{pmatrix} \bar{a}^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -b/\bar{a} & 1 \end{pmatrix}. \tag{3.1.31} \]

Based on the fact that (Q3.1.32)
Proposition 3.1.32. For any fixed $z$, a bounded solution of (3.1.3) normalized at $-\infty$ $(+\infty)$ is unique.

The solutions $m$ and $\tilde{m}$ can also be obtained directly from the following special cases of equation (3.1.5) for $\text{Im} z \neq 0$.

$$m(x, z) = I + \int_{x_0 + \text{sgn}(\text{Im} z)}^{x} e^{i(x-y)z} \text{ad}_q q(y)m(y, z) \, dy$$

where $x_0 = -\infty$ for the (11), (12), (22) entries, $x_0 = +\infty$ for the (21) entry, and

$$\tilde{m}(x, z) = I + \int_{x_0 + \text{sgn}(\text{Im} z)}^{x} e^{i(x-y)z} \text{ad}_q q(y)\tilde{m}(y, z) \, dy,$$

where $x_0 = +\infty$ for the (11), (21), (22) entries, $x_0 = +\infty$ for the (12) entry.

These two equations are not of Volterra type as the lower limits vary for different entries. However, their integral operators are compact and analytic for $z \in \mathbb{C} \setminus \mathbb{R}$ (Q3.1.12a). Hence by analytic Fredholm theorem, solutions $m$ and $\tilde{m}$ are meromorphic in $\mathbb{C} \setminus \mathbb{R}$.

Proposition 3.1.35. Denote by $P$ the set of poles of $m$. Then $P$ is also the set of poles for $\tilde{m}$ and the set of zeros for $\{a, \tilde{a}\}$. Furthermore, any $z' \in P$ has the same multiplicity as a pole of $m$, as a pole of $\tilde{m}$, or as a zero of $\{a, \tilde{a}\}$.

Proof. Let us first assume $\text{Im} z' > 0$. By (3.1.24), the multiplicity of $z'$ as a pole of $m$ cannot be greater than that of $1/a$. But if the former is less than the latter, then $M_1 = m_1(z')$ must vanish identically at $z'$ and this is impossible viewing (3.1.6). Hence they are equal. The rest of the proposition can be proved in a similar manner. \hfill \Box

It seems that it is easier to derive the scattering theory of these two equations. However, it is hard to obtain estimates as good as those obtained from the Volterra integral equations (3.1.6).

Let us first assume that $a(z)$ and $\tilde{a}(z)$ vanish nowhere (including on the boundary). This happens, for instance, when $q$ has certain symmetries (see below) or when $\|q\|_{L^1} < 1$ by (3.1.33)-(3.1.34). In this case, $m$ and $\tilde{m}$ have no singularities (recall $\det m = \det \tilde{m} = 1$), and we may use either the $H^1$ RH problem $(m, e^{i z z} \text{ad}_q v, \mathbb{R})$ or the $H^1$ RH problem $(\tilde{m}, e^{i z z} \text{ad}_q \tilde{v}, \mathbb{R})$ for the inverse scattering transform.

We call $v_\pm$ ($\tilde{v}_\pm$) the scattering data. The inverse scattering transform is to obtain the potential $q$ from the scattering data. The procedure is the same as that of the direct method described in Section 1.3 (Example 1.3) through solving the RH problem. More details of the inverse scattering transform will be discussed later.

One also calls $r^{\text{def}} h/a$ and $\tilde{r}^{\text{def}} \tilde{h}/\tilde{a}$ the continuum part of the scattering data and writes the RH problem as $(m, e^{i z z} \text{ad}_q v, \mathbb{R})$, $v^{\text{def}} \begin{pmatrix} 1 - r\tilde{r} & -\tilde{r} \\ r & 1 \end{pmatrix} = v_+^{-1} v_\pm$. Some natural questions are:

Can we use any $r, \tilde{r} \in H^1$ for scattering data? Can we determine $a, \tilde{a}, h, \tilde{h}$ from $r$ and $\tilde{r}$? To answer these questions, we need a winding number constraint satisfied by $1 - r\tilde{r}$.
Proposition 3.1.36. $W_{\tilde{\tau}}(1 - r^{\tilde{\tau}}) = 0$.

Proof. Using the fact that $\det A = 1$ from (3.1.23a), we obtain

$$1 - r^{\tilde{\tau}} = 1 - \frac{bb}{a^2} = \frac{1}{a^2}.$$

This is a GK factorization because of (3.1.23c) and hence the proposition follows. □

Remarks 3.1.38. When the winding number constraint is satisfied, we can solve the scalar RH problem $(1 - r^{\tilde{\tau}}, R)$ to obtain $a$ and $\tilde{a}$ and thereby $\tilde{b}$ and $\hat{b}$. The upper-lower triangular factorizability of $\tilde{v}$ enables us to recover decay of $q$ as $x \to -\infty$ in the inverse problem, while the lower-upper triangular factorizability of $\hat{v} \equiv \hat{b}_-^{-1} \hat{b}_+$ enables us to recover decay of $q$ as $x \to +\infty$. The matrices $\tilde{v}$ and $\hat{v}$ are related by the following formula,

$$\hat{v} = \tilde{a}^2 \tilde{v} \tilde{a}^\sigma.$$

To study the scattering and inverse scattering problems with poles of $m$ (zeros of $a$ and $\tilde{a}$ or eigenvalues of problem (3.1.1)) and spectral singularities, we need to introduce other solutions of equation (3.1.1). In the classical method, for a pole $z_0$ of $m$, one considers the eigenfunction (or possibly cyclic functions for multiple poles). But the classical method only works when there are finitely many poles. Besides, for more general $n \times n$ problems, the poles of $m$ need not be eigenvalues of the system (hence the scattering theory is richer than the spectrum theory). Our method is based on utilizing certain unbounded solutions of system (3.1.3) given below.

Let $x_0 \in \mathbb{R}$ be such that $\|q\|_{L^1((-\infty, x_0])} < 1$. Using (3.1.33), we have a bounded solution $m^{(0)}$ normalized as $x \to -\infty$ for the potential $q \chi_{(-\infty, x_0)}$. This solution does not have poles or spectral singularities. Also define a solution $m^{(1)}$ for $q$ by the Volterra equation

$$m^{(1)} = I + \int_{x_0}^x e^{i(x-y)z}d\sigma q(y)m^{(1)}(y, z)dy,$$

and another solution for $q$.

$$m^{(2)}(x, z) = m^{(1)}(x, z)e^{i(x-x_0)z}d\sigma m^{(0)}(x_0, z).$$

This solution agrees with $m^{(0)}$ at $x = x_0$ and hence at any $x \leq x_0$ by the uniqueness of solutions of initial value problems for ODEs and the fact that $q = q \chi_{(-\infty, x_0)}$ for $x \leq x_0$. Thus $m^{(2)}$ is normalized as $x \to -\infty$ (but it can be unbounded as $x \to +\infty$). Since $m^{(1)}$ is entire in $z$ and $m^{(0)}(x, \cdot) - I \in A H^1(\mathbb{C} \setminus \mathbb{R})$, $m^{(2)}(x, \cdot) - I \in A H^1(\mathbb{C} \setminus (\mathbb{R} \cup N))$ where $N$ is a disk centered at $\infty$.

An (unbounded) solution $\tilde{m}^{(2)}$ normalized as $x \to +\infty$ can be obtained analogously.

Let us first consider the case when $m$ has no spectral singularities and hence has only a finite number of poles. We denote this set of poles by $P$ and define $v^{(2)}_{z_0}(z)$ for $z_0 \in P$ by

$$m^{(2)}(x, z) = m(x, z)e^{izz}d\sigma v^{(2)}_{z_0}(z).$$
Proposition 3.1.43.
(3.1.43a) \( v_z^{(2)} \in V_z \) (see Section 1.4).
(3.1.43b) \( v_z^{(2)} - I \) is strictly lower/upper triangular if \( \text{Im} \ z' \geq 0 \).

Proof. The first statement is obvious. The second statement follows from the fact that \( e^{iz\hat{z}ad\sigma}v_z^{(2)}(z) \rightarrow I \) as \( z \rightarrow -\infty \) deduced from (3.1.43b). \( \Box \)

Now \( m \) is the solution of the RH problem

\[
\{ m, e^{iz\hat{z}ad\sigma}v(z), z \in \mathbb{R}, e^{iz\hat{z}ad\sigma}v_z^{(2)} \in V_{z'}, z' \in P \}.
\]

where \( m \) satisfies (1.1.2) on \( \mathbb{R} \) and (1.4.1b). This RH problem can be converted into a standard RH problem with the contour

\[
\Gamma^{\text{def}} = \mathbb{R} \cup \bigcup_{z' \in P} S_{z'},
\]

where \( S_{z'} \) is a small counterclockwise oriented circle centered at \( z' \in P \).

A further reduction can be done as follows. We decompose \( \tilde{v}_z - I \) into the sum of the singular part \( u \) and the regular part \( w \) in its Laurent expansion, and set \( v_z = I + u \). By (3b), \( me^{iz\hat{z}ad\sigma}v_z^{(2)} \) is analytic at \( z' \) if and only if \( me^{iz\hat{z}ad\sigma}v_z \) is. Hence we have an equivalent RH problem

\[
\{ m, e^{iz\hat{z}ad\sigma}v(z), z \in \mathbb{R}, e^{iz\hat{z}ad\sigma}v_z^{(2)} \in V_{z'}, z' \in P \}.
\]

and we will call \( \{ v_{z'}, z' \in P \} \) the discrete part of the scattering data. Again, there is a winding number constraint on this RH problem.

Proposition 3.1.47. Counting the multiplicities of poles, \( W_{\mathbb{R}}(1-\tau) = \text{number of poles of } \{ v_{z'}, \text{Im } z' < 0 \} - \text{number of poles of } \{ v_{z'}, \text{Im } z' > 0 \} \).

Proof. By (3.1.37) and proposition 3.1.15, \( W_{\mathbb{R}}(1-\tau) = \text{number of zeros of } a - \text{number of zeros of } a = \text{number of poles of } m, \text{Im } z < 0 - \text{number of poles of } m, \text{Im } z > 0 \) = number of poles of \( \{ v_{z'}, \text{Im } z' < 0 \} - \text{number of poles of } \{ v_{z'}, \text{Im } z' > 0 \} \). Here we count the multiplicities of zeros and poles. \( \Box \)

Let us consider finally the general case, which allows arbitrary numbers of poles and spectral singularities. By (3.1.23c), \( a \) and \( \hat{a} \) cannot have zeros near \( z = \infty \).
Hence we can use \( m \) near \( z = \infty \) and use \( m^{(2)} \) elsewhere. Let \( S_{\infty} \) be a circle centered at \( \infty \) such that \( m \) has no singularities inside the circle. Set \( \Gamma = \mathbb{R} \cup S_{\infty} \) as in the following figure.

Figure 3.1b

where \( \Omega_+ = \Omega_1 \cup \Omega_4, \Omega_- = \Omega_2 \cup \Omega_3 \). Define \( m = m \) on \( \Omega_1 \cup \Omega_2 \), \( m = m^{(2)} \) on \( \Omega_3 \cup \Omega_4 \), and \( e^{i \frac{\pi}{2} \sigma_3} v = m_+^{-1} m_- \). We call a matrix function \( s \) an auxiliary scattering matrix if

\( s - I \in \mathcal{A} H^1(\Omega) \).

The jump matrix \( \nu \) is characterized in the following.

**Theorem 3.1.49.**

\( \text{(C3.1.50a)} \) The matrix \( \nu \) admits a triangular factorization \( \nu = \nu_- ^{-1} \nu_+ \), where \( \nu_{\pm} - I \in H^1(\partial \Omega_{\pm}) \). \( \nu_+ \mid_{\partial \Omega_1} - I \) (\( \nu_- \mid_{\partial \Omega_2} - I \)) is strictly lower (upper) triangular.

\( \text{(C3.1.50b)} \) There exists an auxiliary scattering matrix \( s \) such that \( s^{-1} \nu_{\pm} = \nu_{\pm} ^{-1} \nu_{\pm} \) for some invertible matrices \( \nu_{\pm} \in I + H^1(\partial \Omega_{\pm}) \) with \( \nu_{\pm} ^{-1} \nu_{\pm} \) having opposite triangularities of \( \nu_{\pm} \).

\( \text{(C3.1.50c)} \) The RH problem \( (e^{i \frac{\pi}{2} \sigma_3} \nu, \Gamma) \) is solvable for all \( \nu \in \mathbb{R} \).

For the convenience of the reader, the triangularity of \( \nu_\pm \) and \( \nu_\pm ^{-1} \nu_\pm \) is indicated in the following figure.

Figure 3.1c

Conditions (C3.1.50ab) are written in the form that is easy to use in the inverse problem but hard to check. The following proposition gives an equivalent set of conditions.

**Proposition 3.1.51.** The set of conditions \( (C3.1.50ab) \) is equivalent to the following set of conditions,

\( \text{(C3.1.52a)} \) \( \nu \in H^1(\partial \Omega_-) \cdot H^1(\partial \Omega_+) \), \( \det \nu = 1 \).
(C3.1.52b) The entry $v_{22}(z) = 1$ for $|z| > |S_\infty|$ on the line. where $|S_\infty|$ denotes the radius of $S_\infty$, $v_{11}(z) = 1$ for $|z| < |S_\infty|$ on the line, and $v(z) - I$ is strictly lower/upper triangular for $z \in S_\infty$. $\text{Im } z \gtrless 0$.

(C3.1.52c) There exists an auxiliary scattering matrix $s$ with $s = I$ in $\Omega_1 \cup \Omega_2$ such that $(s^{-1} v s_+)_11(z) \neq 0$ for $|z| > |S_\infty|$ on the line. $(s^{-1} v s_+)_22(z) \neq 0$ for $|z| < |S_\infty|$, and $(s^{-1} v s_+)(z)$ is upper/lower triangular for $z \in S_\infty$. $\text{Im } z \gtrless 0$.

(C3.1.52d–winding number constraint) $W_T(\Delta) = 0$, where $\Delta(z)$ is the diagonal factor in the (upper/lower)-diagonal-(upper/lower) triangular factorization of $(s^{-1} v s_+)(z)$ for $|z| \gtrless |S_\infty|$ on the line. and is the diagonal part of $(s^{-1} v s_+)(z)$ for $z \in S_\infty$.

(3.1.52e) $(\text{C3.1.50c})$

It will become clear in the following proof that $W_T(\Delta)$ in (C3.1.52d) is well defined and the winding number constraint is invariant under different choices of $s$.

**Proof.** In this proof, we denote by $\hat{s}$ the matrix $s$ in (C3.1.52d) to avoid confusion with that in (C3.1.50b).

(C3.1.50abc)⇒(C3.1.52abcde) part:

(C3.1.52ab) follow directly from (C3.1.50a). To show (C3.1.52c), simply set, $\hat{s} = s$ for $|z| < |S_\infty|$ and $\hat{s} = I$ for $|z| > |S_\infty|$ and this $\hat{s}$ works for the $s$ in (C3.1.52c).

To show (C3.1.52d), let $\Delta = I + (z) = \delta_\infty^{-1} \delta_\infty$. For instance, for $z \in (|S_\infty|, \infty)$.

$$
\hat{s}^{-1} v s_+ = \delta_\infty^{-1} s^{-1} v s_+ \delta_\infty = \delta_\infty^{-1} \begin{pmatrix} 1 & 0 \\ \ast & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \ast & 1 \end{pmatrix} \delta_+ = \begin{pmatrix} 1 & 0 \\ \ast & 1 \end{pmatrix} \delta_\infty^{-1} \delta_\infty \begin{pmatrix} 1 & 0 \\ \ast & 1 \end{pmatrix}
$$

and for $z \in \text{the upper semi-circle}$,

$$
\hat{s}^{-1} v s_+ = \delta_\infty^{-1} s^{-1} v s_+ \delta_\infty = \delta_\infty^{-1} \begin{pmatrix} 1 & 0 \\ \ast & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \ast & 1 \end{pmatrix} \delta_+ = \delta_\infty^{-1} \delta_\infty \begin{pmatrix} 1 & 0 \\ \ast & 1 \end{pmatrix}
$$

Thus $W_T(\Delta)$ is well defined and equal to 0.

(C3.1.52a–e)⇒(C3.1.50a–e) part:

To show (C3.1.50a), we need to define $v_+$ on $\partial \Omega_1 \cup \partial \Omega_4$ and $v_-$ on $\partial \Omega_2 \cup \partial \Omega_3$ as in figure 3.1b so that all conditions in (C3.1.50a) are satisfied.

On $\mathbb{R}$ we define $v_\pm$ by the (upper/lower)-identity--(lower/upper) triangular factorization $v_\pm = v_\pm^{-1} v_\pm$ for $|z| \gtrless |S_\infty|$. Existence of such a factorization is guaranteed by (C3.1.52b). We then define $v_\pm$ on the upper/lower semi-circle such that $v_\pm - I$ is smooth, strictly lower/upper triangular and matches $v_\pm - I$ on $\mathbb{R}$ from the sides $|z| > |S_\infty|$ at the points $\pm |S_\infty|$. Thus $v_\pm$ on the upper/lower semi-circle must be equal to $v_\pm - v_\pm^{-1}$ in order that $v = v_\pm^{-1} v_\pm$. Now clearly from the definition that

$$
v_+ \mid \sigma_0, -I \in H^1(\partial \Omega_1), \quad v_- \mid \sigma_0, -I \in H^1(\partial \Omega_2),
$$

and both have the required triangularity. Since $v_+ \mid \sigma_0, -I$ and $v_- \mid \sigma_0, -I$ also have required triangularity, and are $H^1$ away from $\pm |S_\infty|$, the only thing left to be proven is that they are continuous at $\pm |S_\infty|$. Let us only check this continuity at $|S_\infty|$. We denote by, as in the figure below, $v_1, v_2, v_3, v_4$ the limiting values of $v$ at $|S_\infty|$ started from the ray towards east and ordered counterclockwise.
Also denote by $v_{3\pm}$ and $v_{3\pm}$ the limiting values at $|S_\infty|$ of $v_1^\pm$ and $v_3^\pm$, respectively. The matching conditions which we need to show are

$$v_{3+} = \lim_{z \to |S_\infty|} v_+ |_{S_\infty} = \lim_{z \to |S_\infty|} (v_- v) |_{S_\infty} = v_1 - v_4$$

and

$$v_{3-} = \lim_{z \to |S_\infty|} v_- |_{S_\infty} = \lim_{z \to |S_\infty|} (v_+ v^{-1}) |_{S_\infty} = v_1 + v_2^{-1}.$$

where $S^\infty_\pm$ denotes the upper/lower semi-circle.

(C.1.52a) implies that

$$v_1 v_2^{-1} v_3 v_4^{-1} = I.$$

We have

$$v_3^{-1} v_2^{-1} v_3 = v_2 v_3^{-1} v_4 = v_2 v_3^{-1} v_1 - v_4.$$

Now (1) and (2) follow from the uniqueness of the triangular factorization of $v_3$.

We show next that $\Delta \in H^1(\partial \Omega_\pm) \cdot H^1(\partial \Omega_\pm)$ so that $W^\pm(\Delta)$ is well defined. Noticing that (C.3.1.52c) guarantees the existence of $\Delta$ as an $H^1$ function away from the two self-intersections (and $\det \Delta = 1$ by definition), we only need to verify the match conditions for $\Delta$ at the self-intersections. We will only verify the matching condition at $|S_\infty|$. Let us denote the limiting values of $\Delta^{-1}$ and $\Delta$ at $|S_\infty|$ in analogy to those for $v_1, v_2, v_3, v_4$ and $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ respectively. Then condition (C.3.1.52c) says

$$v_1 v_2^{-1} = v_3 v_4^{-1}.$$

Using (C.3.1.52c), we have the following expression for (3.1.53):

$$\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \Delta_1 \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \Delta_2^{-1} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = \Delta_4 \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \Delta_5^{-1} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix},$$

and this can be written in turn as

$$\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \Delta_1 \Delta_2^{-1} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \Delta_4 \Delta_5^{-1} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

By the uniqueness of the lower-diagonal-upper triangular factorization, we must have

$$\Delta_1 \Delta_2^{-1} = \Delta_4 \Delta_5^{-1}.$$
But this is precisely the desired matching condition for $\Delta$. To show that (C3.1.52d) is independent of the choices of $\tilde{s}$, we first use the final part of (C3.1.52c) and the fact that $\tilde{s} = I$ for $|z| > |S_\infty|$ to see that $\tilde{s}^{\pm 1} \psi$ is upper/lower triangular on $S_\pm^\infty$. It is then clear that any other auxiliary scattering matrix must have the form $\tilde{s}(z)H(z)$ for some upper/lower triangular matrix function $H$ for $\text{Im} z \geq 0$. $H \in I + A H^1(\Omega)$ and $\det H = 1$. It is easily verified that with this new auxiliary scattering matrix, $\Delta$ is replaced by $\Delta^\prime = h^{-1}_- \Delta h_+$, where $h$ is the diagonal part of $H$, and hence $W_\tau(\Delta) = W_\tau(\Delta^\prime)$. Finally, by condition (C3.1.52d), $\Delta$ admits a GK factorization

$\Delta = \delta^{-1}_- \delta_+$

with $\delta(\infty) = 1$, and it is straightforward to check that

$\psi \overset{\text{def}}{=} \delta_- \tilde{s}_-^{-1} \psi \delta_+^{-1}$

satisfies (C3.1.50c) with $s = \tilde{s} \delta^{-1}$. \hfill $\square$

Proof of Theorem 3.1.49. By Proposition 3.1.51, we only need to show (C3.1.52a–e). (C3.1.52a) and (C3.1.52e) follow directly from the forward scattering problem.

The first part of (C3.1.52b) follows from the actual construction of $m$ from the potential $q$ for $|z| > |S_\infty|$, and of $m^{(2)}$ from the potential $q|_{\Omega \setminus (-\infty, x_0)}$ for $|z| < |S_\infty|$. The second part of (C3.1.52a) follows simply from the normalization condition $m(\infty) = I$.

To show (C3.1.52c) and (C3.1.52d), let $\tilde{m}$ denote a solution normalized at $x = +\infty$, obtained in a similar manner as $m$. Then

$s \overset{\text{def}}{=} e^{-izx}z \ m_0^{-1} \tilde{m}$

gives an auxiliary matrix for (C3.1.48). The jump matrix $\tilde{\psi}$ for $\tilde{m}$ is related to $\psi$ by $\tilde{\psi} = s^{-1} \psi$, and hence (C3.1.52c) is satisfied. One also checks that $\Delta = \delta^{-1}_- \delta_+$ as in the first part of the proof and hence (C3.1.52d) is satisfied.

Two most important symmetry reductions in applications to integrable systems are:

$\hat{Q} = -\hat{Q}$;

$\hat{Q} = \hat{\Theta}$.

An interesting fact is that in these two cases, the RH problems are always solvable. We saw in Section 1.2 that for the case of the nonlinear Schrödinger equation these two symmetries correspond to the defocusing case and the focusing case respectively.

Let us first consider the case $\hat{Q} = -\hat{Q}$. Rewriting system (3.1.1) in the form

$(-i \sigma \partial_z + i \sigma q) \psi = z \psi$.

we see that it is a self-adjoint problem and hence there are no nonreal eigenvalues. By (Q3.1.20a), $a$ and $\bar{a}$ have no nonreal zeros. Also for real $z$, system (3.1.1) has the symmetry

$\psi(z) = \sigma_1 \overline{\psi(\bar{z})} \sigma_1$.
which induces the symmetry on \( A = \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} \).

(3.1.61) \quad A = \sigma_1 \overline{A} \sigma_1 \quad \Leftrightarrow \quad \bar{a} = \bar{a}, \quad \bar{b} = \bar{b}, \quad (\bar{r} = r).

Thus using \( \det A = 1 \).

(3.1.62) \quad |a|^2 - |b|^2 = 1.

This implies that for real \( z \),

(3.1.63) \quad |a|, |\bar{a}| \geq 1, \quad 1 - |r|^2 = 1/|a|^2 > 0.

We conclude that with the symmetry \( \tilde{Q} = -\bar{Q} \), the are no poles or spectral singularities and the RH problem is \((v, \mathbb{R})\) with \( v = \begin{pmatrix} 1 - |r|^2 & -\bar{r} \\ r & 1 \end{pmatrix} \). The scattering transform \( Q \mapsto r \) is from \( L^2(\mathbb{R}, (1 + x^2)dx) \) to \( H^1(\mathbb{R}, dz) \cap \{|r| < 1\} \).

Now consider the second symmetry \( \tilde{Q} = \bar{Q} \). We will only work with the case with arbitrary numbers of poles and spectral singularities. Special cases can be treated similarly. One checks directly that if \( \psi(x, z) \) is a fundamental solution at \( z \), then

\[ (\psi^*(x, \bar{z}))^{-1} \]

is a fundamental solution at \( \bar{z} \). Since the normalized bouned solution \( m_0 \) and \( m_1 \) are unique,

(3.1.64) \quad m(x, z)m^*(x, \bar{z}) = I, \quad m^{(0)}(x, z)m^{(0)*}(x, \bar{z}) = I.

The Volterra integral equation (3.1.40) also has unique solution normalized at \( x_0 \). Thus

(3.1.65) \quad m(x, z)m^*(x, \bar{z}) = I.

Using this and the fact that the contour \( \Gamma \) is Schwarz reflection invariant with the orientation, we obtain that the symmetry condition satisfied by \( v \) is

(C3.1.66) \quad v(z) = v^*(\bar{z}).

When the orientation of \( \Gamma \) is not Schwarz reflection invariant, such as in the RH problem (3.1.46), condition (C3.1.66) must be modified accordingly. Notice that with either symmetry, conditions (2.2ab) are satisfied and hence the RH problem is solvable.

### 3.2 Inverse Scattering

In this section, we construct the inverse scattering transform from \( H^1 \) to \( L^2((1 + x^2)dx) \). Since \( \det v = \det \overline{v} = 1 \), by Proposition 1.1.4, any normalized solution is a fundamental solution.

For simplicity, we denote \( w_x = e^{ix\sigma_3}w \) for any matrix function \( w \).

Consider first the simplest case when there are no spectral singularities or poles.

The inverse problem is formulated as the RH problem \((v_x, \mathbb{R})\) where \( v = \begin{pmatrix} 1 - r & -\bar{r} \\ r & 1 \end{pmatrix} \).

Let us assume that this RH problem is solvable for all \( x \). As we mentioned in the
previous section, this is automatic in the two symmetry reductions (3.1.57.58). In
the general case this assumption must be made as part of the characterization
problem in the direct scattering problem. By Proposition 2, this RH problem is
equivalent to the integral equation problem,

\begin{equation}
\mu = I + C_{v_{\pm}} \mu.
\end{equation}

where \( v_{\pm} \) are given in (3.1.26) and \( \mu = m^{-1} \) there. Once this integral equation is
solved, we construct

\[ m = I + C_{v_{\pm}} \mu. \]

and

\begin{equation}
q = -i \text{ad} \sigma m_{\infty,1} = \frac{\text{ad} \sigma}{2\pi} \int_{\mathbb{R}} \mu(v_{+} - v_{-}).
\end{equation}

We want to prove the following

**Theorem 3.2.3.** Under the conditions

(3.2.3a) \( \tau, \rho \in H^{1} \)

(3.2.3b) \( 1 - r^{\rho}(z) \neq 0, \ z \in \mathbb{R}, \ W_{T}(1 - r^{\rho}) = 0. \)

(3.2.3c) The partial indices of \( v_{\pm} \) are zero for all \( x \in \mathbb{R}. \)

we have \( q \in L^{2}((1 + x^{2}) dx). \)

**Lemma 3.2.4.** \( \| (1 - C_{v_{\pm}})^{-1} \|_{L^{2}_{\rho}} \) is bounded on any interval \(( -\infty, c), \ c \in \mathbb{R}. \)

**Proof.** (Q3.2.4) \( \Box \)

**Lemma 3.2.5.** For \( x \leq 0. \)

\[ \| C^{-1}(I - v_{-}) \|_{L^{2}} \leq \frac{1}{\sqrt{1 + x^{2}}} \| r \|_{H^{1}} \| \tilde{\rho} \|_{H^{1}}. \]

**Proof.** We will only estimate \( u_{21} \equiv C^{-}(v_{+} - I) \) in which \( u_{21} \) is the only nonzero
entry. \( C^{-}(I - v_{-}) \) may be estimated in a similar manner. By Fourier transform

\[ r(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i \xi z} \tilde{\rho}(\xi) d\xi. \]

and hence

\begin{equation}
u_{21}(x, z) = C^{-} e^{-iz} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} e^{i(\xi - z) \tilde{\rho}(\xi)} d\xi,
\end{equation}

and

\begin{equation}
\| u_{21}(x, \cdot) \|_{L^{2}(\mathbb{R})} = \left( \int_{-\infty}^{\infty} | \tilde{\rho}(\xi) |^{2} d\xi \right)^{1/2} \leq (1 + x^{2})^{-1/2} \| r \|_{H^{1}}.
\end{equation}
Proof of Theorem 3.2.3. We first estimate for $x \leq 0$.

\begin{equation}
\int_{\mathbb{R}} (1 - C_{v_{x \pm}})^{-1} I (v_{x+} - v_{x-}) \, dz \equiv \int_1 + \int_2 + \int_3,
\end{equation}

where

\begin{align*}
\int_1 &= \int (v_{x+} - v_{x-}), \\
\int_2 &= \int (C_{v_{x \pm}} I)(v_{x+} - v_{x-}) \\
\int_3 &= \int (C_{v_{x \pm}} (1 - C_{v_{x \pm}})^{-1} C_{v_{x \pm}} I)(v_{x+} - v_{x-}) \\
&= \int (C_{v_{x \pm}} (\mu - I))(v_{x+} - v_{x-}).
\end{align*}

We remark that for $q$, the estimate for $\int_2$ is not needed because $\int_2$ is diagonal and ad $\sigma \int_2 = 0$. Clearly, $\int (v_{x+} - v_{x-}) \in L^2((1 + x^2) \, dx)$ by Fourier transform. Using the triangle inequality of $v_{x \pm}$, the fact that $C^+ - C^- = 1$, the Cauchy theorem, and Lemma 3.2.5, we have for $x \leq 0$.

\begin{equation}
|\int_2| = \int \left| (C^+(I - v_{x-}))(v_{x+} - I) + (C^-(v_{x+} - I))(I - v_{x-}) \right| \\
= \int \left| (C^+(I - v_{x-}))C^-(I - v_{x+}) + (C^-(v_{x+} - I))C^-(I - v_{x-}) \right| \\
\leq c(1 + x^2)^{-k}, \text{ some } c > 0.
\end{equation}

Finally, for the estimate of $\int_3$, noticing that by Lemmas 3.2.4-5,

\begin{equation}
\|\mu - I\|_{L^2} = \|(1 - C_{v_{x \pm}})^{-1} C_{v_{x \pm}} I\|_{L^2} \leq c(1 + x^2)^{-1/2}, \text{ some } c > 0,
\end{equation}

we have

\begin{equation}
|\int_3| = \int \left[ (C^+(\mu - I)(I - v_{x-}))(v_{x+} - I) \\
+ (C^-(\mu - I)(v_{x+} - I))(I - v_{x-}) \right] \\
= \int \left[ (C^+(\mu - I)(I - v_{x-}))C^-(I - v_{x+}) \right. \\
\left. + (C^-(\mu - I)(v_{x+} - I))C^+(I - v_{x-}) \right] \\
\leq c(1 + x^2)^{-1}, \text{ some } c > 0.
\end{equation}

For the estimates when $x \geq 0$, as in Remarks 3.1.38, we use the winding number constraint in Proposition 3.1.36 to construct $a$, $\tilde{a}$ and $\tilde{u} = \tilde{a}^w u a^w$ as in (3.1.39). The fact that $\tilde{u}$ admits an opposite triangular factorization compared to $u$, we have parallel results for $x \geq 0$. □

Combining this theorem and the results of previous section, we obtain
Theorem 3.2.12. For \( q \in L^2((1 + z^2)dz) \), the scattering data are completely characterized by conditions (3.2.5abc) in the absence of spectral singularities and poles. In the symmetry reduction case (3.1.57), the scattering-inverse scattering transform is a bijection between \( L^2((1 + z^2)dz) \) and \( H^1(dz) \) with the scattering data characterized by the following conditions.

(3.2.13a) \( \hat{r} = r \in H^1 \).
(3.2.13b) \( |r(z)| < 1 \) for each \( z \in \mathbb{R} \).

In the symmetry reduction case (3.1.58), in the absence of poles and spectral singularities, the scattering data are characterized by the single condition.

(3.2.14) \( \hat{r} = -\hat{r} \in H^1 \).

Now let us consider the case without spectral singularities.

Theorem 3.2.15. The potential \( q \) constructed from the RH problem (3.1.46) belongs to the space \( L^2((1 + z^2)dz) \) if the following conditions are satisfied.

(3.2.16a) = (3.2.3a).

(3.2.16b) For each \( z' \in P \), \( u_{z'}(z) - I \) is strictly lower/upper triangular for \( \text{Im} z \geq 0 \) and has the form \( \sum_{k_{z'} \leq 0} a_{z'}(z - z')^{-1} \), \( a_{z'} \neq 0 \).

(3.2.16c) \( W_{z'} = \sum_{\text{Im} z' < 0} k_{z'} - \sum_{\text{Im} z' > 0} k_{z'} \).

(3.2.16d) The RH problem (3.1.46) has a fundamental solution for all \( x \in \mathbb{R} \).

Proof. The estimates for \( x \leq 0 \) are similar because the discrete part of the RH problem only adds in an exponentially decaying part in the operator \( C_{z \in \mathbb{R}} \) as \( x \to -\infty \). Consider the RH problem for \( a, a \) \( (1 - rf)^{-1} \), \( \mathbb{H}(z - z')^{-k_{z'}} \cdot z' \in P \). The winding number of this RH problem is

\[
W_{z'}(1 - r\hat{r}) + \sum_{\text{Im} z' < 0} k_{z'} - \sum_{\text{Im} z' > 0} k_{z'} = -\sum_{\text{Im} z' < 0} k_{z'} + \sum_{\text{Im} z' > 0} k_{z'} + \sum_{\text{Im} z' < 0} k_{z'} - \sum_{\text{Im} z' > 0} k_{z'} = 0.
\]

Hence there is a fundamental solution \( \hat{a}, \hat{a} \) normalized at \( \infty \). Set for real \( z \).

(3.2.17) \[
\hat{v} = \hat{a}^\sigma \hat{a} \sigma
\]
and

(3.2.18) \[
\hat{v}_{z'} = \begin{cases} 
\begin{pmatrix} 1 \\ \frac{1}{a^{\sigma(z_{z'})\sigma}} \\ 1 
\end{pmatrix} & \text{Im} z' > 0, \\
\begin{pmatrix} 1 \\ 0 \\ \frac{1}{a^{\sigma(z_{z'})\sigma}} 
\end{pmatrix} & \text{Im} z' < 0.
\end{cases}
\]

We make the following assertion
(A3.2.19) The RH problem (3.1.46) has a solution \( \hat{m} \) normalized at \( z = \infty \) if and only if the RH Problem

(3.2.20) \[
(\hat{v}, \mathbb{R}, \hat{v}_{z'}, z' \in P)
\]
has a solution \( \hat{m} \) normalized at \( z = \infty \) with the relation

(3.2.21) \[
\hat{m} = \begin{cases} 
m \hat{a}^\sigma, & \text{Im} z > 0, \\
m \hat{a}^{-\sigma}, & \text{Im} z < 0.
\end{cases}
\]
To prove the "only if" part of the assertion, assume that $m$ is a solution of (3.1.46) normalized at $z = \infty$ and define $\tilde{m}$ by (3.2.21). Clearly, $\tilde{m}_+ = \tilde{m}_- e^{iz z^* \sigma \sigma_0}$ for $z \in \mathbb{R}$. For $z'$ with $\text{Im} \ z' > 0$.

(3.2.22)
\[
\tilde{m}_e^{iz z^* \sigma \sigma_0} = me^{iz z^* \sigma \sigma_0} \begin{pmatrix} 1 & 0 \\ -\bar{v}_{z'}(z') & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ \alpha(z')_2 \end{pmatrix} = me^{iz z^* \sigma \sigma_0} \begin{pmatrix} a \\ -(av_{z'}) \end{pmatrix} \begin{pmatrix} 1 \\ \alpha(z')_2 \end{pmatrix}.
\]

Since the order of the zero of $a$ at $z'$ is equal to that of the pole of $(v_{z'})_2$, the RHS of (3.2.22) is analytic at $z'$ and so is the LHS. Similarly, $\tilde{m}_e^{iz z^* \sigma \sigma_0}$ is also analytic at $z'$ with $\text{Im} \ z' < 0$. The "if" part follows from (3.2.22) as well.

Now we can use the RH problem (3.2.20) to obtain the estimate for $q$ on the interval $[0, +\infty)$. The arguments are parallel to the early arguments with opposite triangularity.

**Theorem 3.2.23.** In the absence of spectral singularities, the scattering data for $q \in L^2((1 + x^2)dx)$ are characterized by conditions (3.2.16abcd). In the symmetry reduction case (3.1.58), the scattering data are characterized by

(C3.2.24a) $= (2.2.16b)$
(C3.2.24b) $\tilde{q} = \tilde{r} \in H^1$ and $v_{z'}(z)v_{z'}^*(z) = I$.

Now we consider the general case in which arbitrary spectral singularities are allowed. The scattering data are defined through (C3.1.50ab) and the $\tilde{r}^*$ problem for the inverse problem is

(3.2.25)
\[
(e^{iz z^* \sigma \sigma_0}v, \tilde{r} = \mathbb{R} \cup \mathbb{S}_\infty).
\]

**Theorem 3.2.26.** If $v$ is such that the RH problem (3.2.25) is solvable for $x \in \mathbb{R}$, then $q \in L^2((1 + x^2)dx)$. Here

(3.2.27)
\[
q = -i \text{ad} \sigma m_{\infty, 1} = \frac{\text{ad} \sigma}{2\pi} \int_{\Gamma} \mu e^{iz z^* \sigma \sigma_0} (v_+ - v_-).
\]

**Proof.** As before, we proceed by showing the estimates for $x \leq 0$ first. To simplify our arguments, we make the following reductions of the RH problem.

First choose functions $w \in A(\bar{C} \setminus \Gamma)$ such that

(3.2.28a) $w_\pm \in R(\partial \Omega_\pm)$ and $w_\pm - I = O(z^{-2})$ as $z \to \infty$.
(3.2.28b) $w_\pm$ has the same triangularity as $v_\pm$.
(3.2.28c) $w_\pm(z) = v_\pm(z)$ for $z \in \mathbb{R} \cap S_\infty$.

Thus we can write

\[
v = w_1 - (v_+ w_+^{-1})^{-1}(v_+ w_+^{-1})w_+ \equiv w_1^{-1} v_+ w_+^{-1} w_+ \equiv w_1^{-1} v w_1^{-1},
\]

where $v_\pm$ vanishes at $\mathbb{R} \cap S_\infty$. Clearly for $x \leq 0$, $v_x$ admits a GK factorization $m_{-1}m_+$ if and only if $v_x$ admits a GK factorization $m_{-1}m_+$ with $m = me^{iz z^* \sigma \sigma_0}$ because $e^{iz z^* \sigma \sigma_0} \in AL^\infty(\bar{C} \setminus \Gamma) \cap AL^2(\bar{C} \setminus \Gamma)$ guaranteed by (3.2.28ab).

Thus

\[
q = -i \text{ad} \sigma m_{\infty, 1} = -i \text{ad} \sigma m_{\infty, 1} - i \text{ad} \sigma e^{iz z^* \sigma \sigma_0} w_{\infty, 1} = -i \text{ad} \sigma m_{\infty, 1}.
\]
This says \( u \) makes no contribution for \( x \leq 0 \) and therefore we may work with \( v \) which satisfies

\[(3.2.29) \ v_\pm(z) = I \text{ for } z \in \mathbb{R} \cap S_\infty. \]

We then augment the contour by adding in an ellipse as in the following figure.

![Figure 3d](image)

Noting that the orientation of the line segment \((-|S_\infty|, |S_\infty|)\) has been reversed.

we redefine \( v_\pm \) as follows.

\[(3.2.30a)\ v_\pm = I \text{ on the added ellipse.} \]

\[(3.2.30b)\ v_\pm(z) \text{ is the lower/upper triangular factor in the lower–upper triangular} \]

\factorization\ of \( v \) (\( v^{-1} \)) on \( \mathbb{R} \) for \( |z| > |S_\infty| \). \( |z| < |S_\infty| \).

\[(3.2.30c)\ For \ z \in S_\infty, \ v_\pm(z) = I \text{ for } \Im z \leq 0 \text{, and } v_\pm(z) = v(z) \text{ for } \Im z \geq 0. \]

By the assumption \((3.2.29)\), the newly defined \( v_\pm - I \) is in \( H^1(\partial \Omega_\pm) \) and is strictly lower/upper triangular. This is why we augmented the contour.

\textbf{Lemma 3.2.31.} \( \| (1 - C_{v_{+\pm}})^{-1} \|_{L^2(\Gamma)} \) is bounded on any interval \((-\infty, c), c \in \mathbb{R} \).

\textit{Proof.} (Q3.2.4) \( \square \)

\textbf{Lemma 3.2.32.} For \( x \leq 0 \),

\[(3.2.33a)\ \| C_{\Gamma_+}^{-1}(I - v_\pm) \|_{L^2} \leq \frac{c}{\sqrt{1 + z^2}} \| v_\pm - I \|_{H^1} \]

\[(3.2.33b)\ \| C_{\Gamma_-}^{-1}(v_\pm - I) \|_{L^2} \leq \frac{c}{\sqrt{1 + z^2}} \| v_\pm - I \|_{H^1} \]

where \( \Gamma_\pm = \mathbb{R}(\{ \Im z \geq 0 \} \cap \Gamma) \) and \( \Gamma' = S_\infty \cup \{ \Im z \geq 0 \} \) or \( \mathbb{R} \). Also

\[(3.2.33c)\ \| (C_{v_{+\pm}} I)^2 \|_{L^2(\Gamma)} \leq \frac{c}{\sqrt{1 + z^2}} (\| v_\pm - I \|_{H^1} + \| v_\pm - I \|_{H^1}) \]

\textit{Proof.} We will only estimate \( C_{\Gamma_+}^{-1}(v_\pm - I) \) and \( C_{\Gamma_-}^{-1}(I - v_\pm) \) may be estimated in a similar manner. We write

\[(3.2.34)\ u' = C_{\Gamma_+}^{-1}(v_\pm - I),\]

\( u'' = C_{S_\infty \cup \{ \Im z > 0 \} \to \Gamma_+}(v_\pm - I) \).
Since $u'$ may be rewritten as
\[ u' = -C_{\mathbf{R}^- \times \Gamma_-} C_{\mathbf{R}^-} (v_{z^+} - I),\]
by the boundedness of $C_{\mathbf{R}^- \times \Gamma_-}$ and Lemma 3.2.5.
\[ \|u'\|_{L^2(\mathbf{R}^+)} \leq \frac{c}{\sqrt{1 + x^2}} \|v_+ - I\|_{H^1}. \]

For the estimate for $u''$, we only need to show that
\[ (3.2.35a) \quad \|v_{z^+} - I\|_{L^2(\mathbf{R}^+)} \leq \frac{c}{\sqrt{1 + x^2}} \|v_+ - I\|_{H^1}. \]

This follows directly from a scaling argument (Q3.2.32). Now using the triangularity of $v_{\pm}$, the fact that $\text{supp } v = \subset \Gamma_\pm$, and (3.2.33ab), we obtain
\[ \|C_{\Gamma_+}^{-1} (C_{\Gamma_-}^{-1} (v_{z^+} - I) (I - v_{z^-})) + (C_{\Gamma_+}^{-1} (I - v_{z^-})) (v_{z^+} - I))\|
\[ = \|C_{\Gamma_+} C_{\Gamma_-}^{-1} (C_{\Gamma_-}^{-1} (v_{z^+} - I) (I - v_{z^-})) + (C_{\Gamma_+} (I - v_{z^-})) (v_{z^+} - I))\|
\[ \leq c (1 + x^2)^{-\frac{1}{2}}. \]

We mention here that the following estimate is needed for the estimate of $C_{\Gamma_+}^{-1} (I - v_{z^-})$.
\[ (3.2.35b) \quad \|v_{z^-} - I\|_{L^2(\mathbf{R}^+)} \leq \frac{c}{\sqrt{1 + x^2}} \|v_+ - I\|_{H^1}. \]

\[ \square \]
Let us continue the proof of Theorem 3.2.26
\[ (3.2.36) \quad \int_{\mathbf{R}} (1 - C_{\mathbf{R} \times \mathbf{R}}^{-1}) I (v_{z^+} - v_{z^-}) d\sigma \equiv \int_1 + \int_2 + \int_3 + \int_4, \]
where
\[ \int_1 = \text{ad} \sigma \int (v_{z^+} - v_{z^-}), \]
\[ \int_2 = \int (C_{\mathbf{R} \times \mathbf{R}} I)(v_{z^+} - v_{z^-}), \]
\[ \int_3 = \int ((C_{\mathbf{R} \times \mathbf{R}})^2 I)(v_{z^+} - v_{z^-}), \]
\[ \int_4 = \int (C_{\mathbf{R} \times \mathbf{R}} (1 - C_{\mathbf{R} \times \mathbf{R}})^{-1} (C_{\mathbf{R} \times \mathbf{R}})^2 I)(v_{z^+} - v_{z^-}). \]

We write
\[ (3.2.37) \quad \int = \int_{\mathbf{R}} (v_{z^+} - v_{z^-}) + \int_{\mathbf{R}^+} (v_{z^+} - I) + \int_{\mathbf{R}^-} (I - v_{z^-}). \]
The first integral in (3.2.37) belongs to $L^2((1 + x^2)dx)$ by Fourier transform. By (3.2.29) we can use the integration by part to estimate

$$|\int_{S_{\pm}} (v_\pm - I)| = |x^{-1} \int_{S_{\pm}} \partial_x v_\pm| \leq x^{-1} \int_{S_{\pm}} e^{x \text{Im} z} \partial_z (v_\pm) dz.$$

The integral on the last line is a Laplace type of transform and hence belongs to $L^2((\infty, 0))$. This shows that the second integral of (3.2.37) belongs to $L^2((1 + x^2)dx)$ as well. The estimate for the third integral is similar.

To estimate $\int_2$, using the triangularity of $v_\pm$, the fact that supp $v_\pm = \Gamma_\pm$, the fact that $C^+ - C^- = 1$, the Cauchy theorem, and Lemma 3.2.32. we have for $x \leq 0$,

$$\int_2 = \int_{S_{\pm}} [(C^+(I - v_\pm)) (v_\pm - I) + (C^- (v_\pm - I))(I - v_\pm)]$$

$$= \int_{S_{\pm}} (C^+_{\Gamma_+ - \Gamma_-} (I - v_\pm)) (v_\pm - I) + \int_{S_{\pm}} (C^-_{\Gamma_+ - \Gamma_-} (v_\pm - I)) (I - v_\pm)$$

$$+ \int_{\mathbb{R}} (C^+_{\Gamma_+ - \Gamma_-} (I - v_\pm))(v_\pm - I) + \int_{\mathbb{R}} (C^-_{\Gamma_+ - \Gamma_-} (v_\pm - I))(I - v_\pm)$$

$$= \left( \int_{S_{\pm}} (C^+_{\Gamma_+ - \Gamma_-} (I - v_\pm))(v_\pm - I) + \int_{S_{\pm}} (C^-_{\Gamma_+ - \Gamma_-} (v_\pm - I))(I - v_\pm) \right)$$

$$+ \int_{\mathbb{R}} (C^+_{\Gamma_+ - \Gamma_-} (I - v_\pm))C^-_{\mathbb{R}} (I - v_\pm) + \int_{\mathbb{R}} (C^-_{\Gamma_+ - \Gamma_-} (v_\pm - I))C^+_{\mathbb{R}} (I - v_\pm).$$

The desired estimates for the first two integrals in (3.2.38) follow from (3.2.33ab) and (3.2.35ab), while the estimates for the last two integrals follow just from (3.2.33ab). The estimate for $\int_3$ is similar to that for $\int_2$.

$$\int_3 = \int [(C^+ (C^- (v_\pm - I))(I - v_\pm)(v_\pm - I)]$$

$$+ (C^- (C^+ (I - v_\pm))(v_\pm - I))(I - v_\pm)]$$

$$= \int_{S_{\pm}} (C^+_{\Gamma_- - \Gamma_+} (C^- (v_\pm - I))(I - v_\pm) + \int_{S_{\pm}} (C^-_{\Gamma_- - \Gamma_+} (v_\pm - I))(I - v_\pm)$$

$$+ \int_{\mathbb{R}} (C^+_{\Gamma_- - \Gamma_+} (C^- (I - v_\pm))(v_\pm - I))(I - v_\pm)$$

$$+ (C^-_{\Gamma_- - \Gamma_+} (C^- (v_\pm - I))(I - v_\pm))$$

$$= \int_{S_{\pm}} (C^+_{\Gamma_- - \Gamma_+} (C^- (v_\pm - I))(I - v_\pm) + \int_{S_{\pm}} (C^-_{\Gamma_- - \Gamma_+} (v_\pm - I))(I - v_\pm)$$

$$+ \int_{\mathbb{R}} (C^+_{\Gamma_- - \Gamma_+} (C^- (I - v_\pm))(v_\pm - I))(I - v_\pm) + (C^-_{\Gamma_- - \Gamma_+} (I - v_\pm))C^+_{\mathbb{R}} (I - v_\pm).$$
The first two integrals on the last line of the formula above satisfy the desired estimate by (3.2.33ab) and (3.2.35ab), and the last two integrals satisfy the desired estimate by (3.2.33ab) alone.

Finally, for the estimate of $\int_4$, denoting

$$w = (1 - C_{v_{1\pm}})^{-1}(C_{v_{1\pm}})^2 I$$

and using (3.2.33abc) and (3.2.35ab), we have

$$|\int_4| = |\int_{S_1^\infty}(C^+ w(I - v_{1\pm}))(v_{1\pm} - I) + (C^- w(v_{1\pm} - I))(I - v_{1\pm})|$$

$$+ |\int_{S_1^\infty}(C^+ w(v_{1\pm} - I))(I - v_{1\pm})|$$

$$+ |\int_{S_1^\infty}(C^- w(v_{1\pm} - I))(I - v_{1\pm})|$$

$$= |\int_{S_1^\infty}(C^+ w(I - v_{1\pm}))(v_{1\pm} - I)$$

$$+ |\int_{S_1^\infty}(C^- w(v_{1\pm} - I))(I - v_{1\pm})|$$

$$+ |\int_{S_1^\infty}(C^- w(v_{1\pm} - I))(I - v_{1\pm})|$$

$$\leq c(1 + x^2)^{\gamma}, \text{ some } c > 0.$$  

The estimate on the interval $[0, \infty)$ can be analogously obtained from (C3.1.50b) by considering the RH problem $(\tilde{m}, \tilde{m}, \Gamma)$ with $\tilde{m} = ms_{1\pm}$. □

Combining this theorem and the results of previous section, we obtain

**Theorem 3.2.40.** For $q \in L^2((1 + x^2)dx)$, the scattering data are completely characterized by conditions (3.2.5abed).

In the symmetry reduction case (3.1.58), conditions (C3.1.52de) are implied by the symmetry.

### 3.3. Higher Orders of Decays and Regularities

In the previous two sections, we studied the scattering and inverse scattering transforms for the potential space $L^2((1 + x^2)dx)$ and the scattering space $H^1$. However, the associated integrable nonlinear evolutions usually do not stay in $L^2((1 + x^2)dx)$. For example, the NLS evolution in the scattering space has the form $e^{itz^2}\delta^x$ which obviously does not stay in $H^1$ because a multiplier $z$ is created when you differentiate the expression. A right space for this evolution is clearly $H^{1,1}$ which denotes the (weighted) Sobolev space with the norm

$$\|f\|_{H^{1,1}} \overset{\text{def}}{=} (\|f\|_{L^2}^2 + \|\partial_x f\|_{L^2}^2 + \|f\|_{L^2}^2)^{\frac{1}{2}}.$$
In this section, we work with the $L^2$-Sobolev space $H^{j,k}$ which has the norm

$$
\|f\|_{H^{j,k}} \overset{\text{def}}{=} (\|f\|_{L^2}^2 + \|Q^k f\|_{L^2}^2 + \|f^{(j)}\|_{L^2}^2)^{\frac{1}{2}}.
$$

The reader may extend the theory to other $L^2$-Sobolev spaces. We will show that very much like the Fourier transforms the scattering and inverse scattering transforms give a bijection between the space $H^{j,k}$ to the space $H^{k,j}$ for any $j \geq 0, k \geq 1$.

**Proposition 3.3.1.** Assuming $q \in H^{0,k}$, $k \geq 1$, then $A - I \in H^k$.

**Proposition 3.3.2.** Assuming $q \in H^{l,1}$, $l \geq 0$, then $b, \bar{b} \in H^{0,l}$.

**Proof.** The formula

$$
b = \int \hat{Q}e^{iyz}m_{11}^{(-)} = \int \hat{Q}e^{iyz}(m_{11}^{(-)} - 1) + \int \hat{Q}e^{iyz}
$$

will be used to estimate $b$. For $\bar{b}$, the proof is similar and omitted here. Since the second term in the RHS is clearly in $L^2((1 + |z|^{2l})dz)$, we only need to show that $\int \hat{Q}e^{iyz}(m_{11}^{(-)} - 1)$ is in $L^2((1 + |z|^{2l})dz)$.

Denote $K = K_{q,z}$. Noting that $ad \sigma$ is invertible on off-diagonal matrices we compute through integration by parts

$$(KI)(x) = -\sum_{j=0}^{l-1} (iz \cdot ad \sigma)^{-j-1}q^{(j)} + (\mathrm{id} \cdot ad \sigma)^{-l} \int_{-\infty}^{z} e^{i(z-y)^{\cdot}ad \sigma} q^{(l)}(y)dy \equiv h_1 + h_2.
$$

Note that we are only interested in the decay at $z = \infty$, the singularity at $z = 0$ is irrelevant.

We write

$$
m^{(-)} - I = (1 - K)^{-1}KI = h_1 + Kh_1 + (1 - K)^{-1}K^2h_1 + (1 - K)^{-1}h_2 \equiv h_1 + g_1 + g_2 + g_3.
$$

Thus we need to show that

$$
\int \hat{Q}e^{iyz}(h_1 + g_1 + g_2 + g_3)
$$

is in $L^2((1 + |z|^{2l})dz)$. It is obvious that $\int \hat{Q}e^{iyz}h_1$ is in $L^2((1 + |z|^{2l-2})dz)$.

Since $qq^{(j)}$ is diagonal,

$$(K(z^{-j-1}q^{(j)})(x) = z^{-j-1} \int_{-\infty}^{x} e^{i(x-y)^{\cdot}ad \sigma} q(y)q^{(j)}(y)dy = z^{-j-1} \int_{-\infty}^{y} q(\nu)q^{(j)}(\nu)\
$$

It is straightforward to check that $\int \hat{Q}e^{iyz}g_1$ is in $L^2((1 + |z|^{2l+2})dz)$.

Since

$$
q \int_{-\infty}^{x} q^{(j)} \in H^{l-j}(dz)
$$

and is off-diagonal,

$$(K^2(z^{-j-1}q^{(j)})(x) = z^{-j-1} \int_{-\infty}^{x} e^{i(x-y)^{\cdot}ad \sigma} q(y) \int_{-\infty}^{y} q(\nu)q^{(j)}(\nu)du
$$

is in $L^2((1 + |z|^{2l+2})dz) \otimes L^\infty(dx)$ and so is $g_2$. We see that $\int \hat{Q}e^{iyz}g_2$ is also in $L^2((1 + |z|^{2l+2})dz)$.

Finally, $h_2$ is in $L^2((1 + |z|^{2l})dz) \otimes L^\infty(dx)$ and so is $g_3$. Thus $\int \hat{Q}e^{iyz}g_3$ is also in $L^2((1 + |z|^{2l})dz)$.

Now for the forward problem, we have
Theorem 3.3. Assuming $q \in H^{l,k}$, $l \geq 0$, $k \geq 1$, then $\tilde{v}_\pm \in I + H^{k,l}$.

Theorem 2.41. If $v_\pm - I \in H^{l} \cap L^2((1+|z|^2)|dz|)$, then $q \in H^{l}$.

Proof. Using the relation $\partial_z \mu = (iz \text{ad } \sigma + q) \mu$ and the fact that $\text{ad } \sigma$ is a derivation, we have

$\partial_z (\mu e^{iz \sigma} \text{ad } \sigma(v_\pm - v_-)) = (iz \text{ad } \sigma + q)(\mu e^{iz \sigma} \text{ad } \sigma(v_\pm - v_-))$.

Similarly,

$\partial_z^2 (\mu e^{iz \sigma} \text{ad } \sigma(v_\pm - v_-)) = \sum_{j=0}^{h-1} c_j z^j \mu(v_{z^+} - v_{z^-}) + (i \text{ad } \sigma)^h z^h \mu(v_{z^+} - v_{z^-})$,

where $c_j$ is a noncommutative polynomial of $q, q', \ldots, q^{(h-1)}$ without the 0th order term. Thus

$\frac{\text{ad } \sigma}{4\pi} \int \left( \sum_{j=0}^{h-1} c_j z^j \mu(v_{z^+} - v_{z^-}) + (i \text{ad } \sigma)^h z^h \mu(v_{z^+} - v_{z^-}) \right)$.

We first assert that if $h < l$, then $q^{(l)} \in H^1$. Assuming inductively $q^{(l)} \in H^1$ for $j = 0, \ldots, h-1$, we see that $\int z^j \mu(v_{z^+} - v_{z^-}) \in H^1(dx)$ and that the assertion can be proved based on the same kind of estimates as in the proof of Theorem 2.26. Now let $h = l$. Then the first term (the sum in (2.43)) still gives rise to an $H^1$ function. Since $z^h (v_{z^+} - v_{z^-})$ is only in $L^2$. The method in the proof of Theorem 2.26 shows the second term in (2.43) gives rise to an $L^2$ function.

3.4. Time Evolutions and Conservation Laws. As we discussed in section 1.2, the time evolution equations associated to the AKNS system form the NLS hierarchy given by (1.2.?). The time evolution for the scattering data has the form

$e^{i \text{ad } \sigma x} v,$

where $p(z)$ is a polynomial. We assume that $p(z)$ has real coefficients $a_k$ so that the scattering data do not grow as $z \to \infty$. If the degree of $p(z)$ is $n$, then a right space for the scattering data is $H^{1,n-1}$. By the results of the previous section, the corresponding space for the potential is $H^{n-1,1}$.

Theorem 3.4.2. For any initial data $q(\cdot, 0) \in H^{n-1,1}$, the solution (in weak sense) $q(\cdot, t)$ for (1.2.?) exists for $|t| < \epsilon$ for some $\epsilon > 0$.

Furthermore, in the two symmetry reduction cases (3.1.57) and (3.1.58), the solution exists for all $t \in \mathbb{R}$.

Proof. First assume that $v - I$ has compact support. We know from (1) that the RH problem

$e^{i(z \sigma + tp(z)) \text{ad } \sigma x} v$

is solvable for $|x|$ sufficiently large. We also know from the forward problem that the RH problem is solvable at $t = 0$ for any $x \in \mathbb{R}$. Combining these with a continuity argument, we see that the RH problem is solvable for $|t|$ small enough. Clearly $m$ is then real analytic in $x$ and $t$ whenever the RH problem is solvable. Thus the matrix functions $U$ and $V$ in the Lax pair are real analytic in $x$ and $t$ and the Lax generate a solution of (1.2.?) for the initial value problem.

By Remark 1.2.15, the term with the highest derivative (of order $n$) of the evolution equation is linear. Hence an $H^{n-1,1}$ weak solution exists.

In the symmetry reduction cases, the RH problem is solvable for all $x$ and $t$ and therefore the solution exits for all $t \in \mathbb{R}$.
**Theorem 3.4.** Under the hypotheses of Theorem 3.4.2, the solution of (1.2.?) is unique.

Since the matrix $\Delta$ in (3.1.52d) is diagonal, it remains unchanged in the time evolution. Hence it is a conserved quantity in the scattering space and so is the quantity

$$\delta = e^{Ct \log \Delta}.$$ 

Since this conserved quantity is solely defined through the scattering transform of the AKNS-ZS system, $\delta$ is a conserved quantity for the entire hierarchy.
Exercises.

(Q3.1) Consider the Volterra integral equation

\[ m(x, z) = f(x, z) + \int_{-\infty}^{z} K(x, y, z) m(y, z) \, dy, \]

where

(Q3.1.9b) Derive (3.1.9b) from Minkowski's inequality (3.1.9a).

(Q3.1.14) Hardy's inequalities may be understood as that one degree of weight is lost during the integration. Let \( h(x) = \int_{x}^{\infty} f(y) \, dy \) where \( f \in L^1(\mathbb{R}_+) \). Show that

\[ \|Q^p h\|_{L^p(\mathbb{R}_+)} \leq \frac{p}{1 + sp} \|Q^{p-1} f\|_{L^p(\mathbb{R}_+)} \quad p \geq 1, \quad sp + 1 > 0. \]

Hint: First assume that \( f \) is positive, smooth, and has a compact support in \( \mathbb{R}_+ \).

Use integration by parts to show

\[ (1 + sp) \int_{0}^{\infty} (x^{p} h(x)) \, dx = p \int_{0}^{\infty} (x^{p-1} h(x)) \, dx. \]

Then apply the Hölder inequality properly.

(Q3.1.20a) Show that if \( \det M(z, z_0) = 0 \) for some \( z_0, \text{Im} \, z_0 \neq 0 \), then (3.1.1) has a \( L^2 \) solution and hence \( z_0 \) is an \( L^2 \) eigenvalue of (3.1.1).

(Q3.1.12) Prove Proposition 3.1.12.

(Q3.1.12a) Assume \( q \in L^1 \). Show that the integral operators in (3.1.33–3.1.34) are compact and analytic in \( z \) on the space \( \mathcal{C}(\mathbb{R}) \cap L^\infty \).

(Q3.2.4) Let \( v \) be defined as in (C3.1.50) and \( T_{v(x)} \) as in (2.4.11). Show that

(a) \( T_{v(x)} \) is uniformly bounded for \( x \in (-\infty, c) \) for any \( c \in \mathbb{R} \).

(b) Use (2.4.11) to show that \( \|(1 - C_{v(x)}^{-1})\|_{L^2} \) is uniformly bounded for \( x \in (-\infty, c) \) for any \( c \in \mathbb{R} \).

(Q3.2.32) Show (3.3.32)

\[ \|(x \cdot R(\mathbb{R} \cap \{ |z| > 0 \})) \|_{L^2(\mathbb{R} \cap \{ |z| > 0 \})} \leq \frac{c}{1 + x^2} \|v_+ - I\|_{H^1}. \]

Hint: Away from any neighborhoods of \( \{-z_0, z_0\} = \mathbb{R} \cap S_\infty \), the function being estimated decays exponentially as \( x \to -\infty \). Near \( \pm z_0 \), use the scaling \( (x \pm z_0)/t \) to obtain the estimate. Recall the facts that \( v_+(\pm z_0) - I = 0 \) and that the Hölder norm \( \|\cdot\|_{H^1(\mathbb{R} \cap \{ |z| > 0 \})} \) is dominated by the \( H^1 \) norm.

(Q3.2.7) Show by using the Hardy's inequality that, as in 3.2.7,

\[ \|(x)^{1/2} u_{21}\|_{L^2(\mathbb{R} \cap \{ |z| > 0 \})} \leq \|\tau\|_{H^1}. \]

Then use this to show that

\[ \int (\mu - I)(v(x)_{-} - v(x)_{+}) \in L^2((1 + x^2)^{3/2} \, dx). \]
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