Institute for Analysis
PD Dr. Rainer Mandel
Dr. Zoïs Moitier

May 13, 2021

Sobolev spaces: tutorials

Exercise sheet 01

Exercise 1.
The generalized Hölder's inequality: show that for $m \geq 2$ functions $(f_i)_{1\leq i\leq m}$ and $1 \leq p_1, \dots, p_m \leq \infty$, we have

$$\|f_1 \cdots f_m\|_r \leq \|f_1\|_{p_1} \cdots \|f_m\|_{p_m} \qquad \text{for } \frac{1}{r} = \sum_{i=1}^{m} \frac{1}{p_i}.$$

(hint: induction)

Exercise 2.
For a sequence of points $(r_k)_{k\in\mathbb{N}}$ dense in $B_1(0)$ the unit ball of $\mathbb{R}^N$ and $\alpha > 0$, the function

$$f : x \mapsto \sum_{k\in\mathbb{N}} 2^{-k} |x - r_k|^{-\alpha}$$

is in $\mathrm{W}^{1,p}(B_1(0))$ for $\alpha < \frac{N-p}{p}$, but is unbounded in each open subset of $B_1(0)$.

Exercise 3.
For a Lipschitz domain $\Omega \subset \mathbb{R}^N$ and a function $A \in C^1(\Omega; \mathbb{R}^{N\times N})$, find the weak formulation for

$$\begin{cases} -\operatorname{div}(A\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}. \tag{1}$$

Exercise 4.
Let X, Y be Banach spaces and assume that $X_0 \subset X$ is a dense subset. Furthermore assume that there is a linear operator $T_0 : X_0 \to Y$ such that $\|T_0 f\|_Y \leq C \|f\|_X$ for all $f \in X_0$. Show that there is a unique bounded linear operator $T : X \to Y$ such that $T|_{X_0} = T_0$ with operator norm $\leq C$.