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Sobolev spaces: tutorials

Exercise sheet 01 with solution

Exercise 1.
The generalized Hölder's inequality: show that for $m \geq 2$ functions $(f_i)_{1\leq i\leq m}$ and $1 \leq p_1, \dots, p_m \leq \infty$, we have

$$\|f_1 \cdots f_m\|_r \leq \|f_1\|_{p_1} \cdots \|f_m\|_{p_m} \qquad \text{for } \frac{1}{r} = \sum_{i=1}^{m} \frac{1}{p_i}.$$

(hint: induction)

Solution 1.
Step 0: First, the case $r = \infty$ can be treated directly because it implies that $p_1 = \cdots = p_m = \infty$ and

$$\|f_1 \cdots f_m\|_\infty \leq \|f_1\|_\infty \cdots \|f_m\|_\infty$$

comes from the properties of the essential supremum. From now on, we assume that $1 \leq r < \infty$ and the rest of the proof is done in two steps.

Step 1: let's show that for two functions f and g, we have

$$\|f \cdot g\|_r \leq \|f\|_p \cdot \|g\|_q \qquad \text{for } \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

The identity $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ imply that $\frac{r}{p} + \frac{r}{q} = 1$ and $1 \leq \frac{p}{r}, \frac{q}{r} \leq \infty$. We apply the Hölder's inequality to $|f|^r$ and $|g|^r$ with $\frac{p}{r}$ and $\frac{q}{r}$, we obtain

$$\| |f|^r \cdot |g|^r \|_1 \leq \| |f|^r \|_{\frac{p}{r}} \cdot \| |g|^r \|_{\frac{q}{r}}$$

we raise the previous inequality to the power $\frac{1}{r}$ to get

$$\| |f|^r \cdot |g|^r \|_1^{\frac{1}{r}} \leq \| |f|^r \|_{\frac{p}{r}}^{\frac{1}{r}} \cdot \| |g|^r \|_{\frac{q}{r}}^{\frac{1}{r}} \quad \text{which reduces to} \quad \|f \cdot g\|_r \leq \|f\|_p \cdot \|g\|_q .$$

Step 2: We reason by induction. The case $m = 2$ is done by *Step 1*. For $m \geq 3$, we assume that we have the generalized Hölder's inequality for $n \in \{2, \dots, m-1\}$. We set $\frac{1}{p} = \frac{1}{r} - \frac{1}{p_m}$ and $\frac{1}{q} = \frac{1}{p_m}$, the identity $\frac{1}{r} = \sum_{i=1}^{m} \frac{1}{p_i}$ implies that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $1 \leq p, q \leq \infty$. We use *Step 1* on $f = f_1 \cdots f_{m-1}$ and $g = f_m$ with triple (r, p, q), we obtain

$$\|f_1 \cdots f_{m-1} \cdot f_m\|_r \leq \|f_1 \cdots f_{m-1}\|_p \cdot \|f_m\|_{p_m} .$$

If $p = \infty$, we apply *Step 0* and if $p < \infty$, we apply the induction hypothesis on $f_1, \dots, f_{m-1}$ with $\frac{1}{p} = \sum_{i=1}^{m-1} \frac{1}{p_i}$ to get

$$\|f_1 \cdots f_{m-1}\|_p \leq \|f_1\|_{p_1} \cdots \|f_{m-1}\|_{p_{m-1}} \quad \text{and obtain} \quad \|f_1 \cdots f_m\|_r \leq \|f_1\|_{p_1} \cdots \|f_m\|_{p_m} .$$

Therefore, by induction we have proved the generalized Hölder's inequality.

Exercise 2.
For a sequence of points $(r_k)_{k\in\mathbb{N}}$ dense in $B_1(0)$ the unit ball of $\mathbb{R}^N$ and $\alpha > 0$, the function

$$f : x \mapsto \sum_{k\in\mathbb{N}} 2^{-k} |x - r_k|^{-\alpha}$$

is in $\mathrm{W}^{1,p}(B_1(0))$ for $\alpha < \frac{N-p}{p}$, but is unbounded in each open subset of $B_1(0)$.

Solution 2.
We note $f_k : x \mapsto 2^{-k} |x - r_k|^{-\alpha}$, first we show that $f_k \in \mathrm{W}^{1,p}(B_1(0))$. We compute

$$\begin{aligned}
\|f_k\|_p^p &= 2^{-kp} \int_{B_1(0)} |x - r_k|^{-\alpha p} \,\mathrm{d}x \\
&= 2^{-kp} \int_{B_1(r_k)} |y|^{-\alpha p} \,\mathrm{d}y && (y = x - r_k) \\
&\leq 2^{-kp} \int_{B_2(0)} |y|^{-\alpha p} \,\mathrm{d}y && (B_1(r_k) \subset B_2(0)) \\
&\leq 2^{-kp} A_N \int_0^2 r^{-\alpha p}\ r^{N-1} \,\mathrm{d}r && \left(A_N = \int_{\mathbb{S}^{N-1}} 1 \,\mathrm{d}\sigma(\omega)\right) \\
&\leq 2^{-kp}\ A_N \frac{2^{N-\alpha p}}{N - \alpha p} && (N - \alpha p > p \geq 1)\,. \qquad (1)
\end{aligned}$$

For $1 \leq i \leq N$, we compute

$$\partial_i f_k(x) = \partial_i \left[\left(\sum_{j=1}^N \left(x_j - (r_k)_j \right)^2 \right)^{-\frac{\alpha}{2}} \right] = -\alpha \frac{x_i - (r_k)_i}{|x - r_k|^{\alpha+2}}$$

then we have

$$\begin{aligned}
\|\partial_i f_k\|_p^p &= 2^{-kp} \int_{B_1(0)} \left| -\alpha \frac{x_i - (r_k)_i}{|x - r_k|^{\alpha+2}} \right|^p \mathrm{d}x \\
&= 2^{-kp} \alpha^p \int_{B_1(r_k)} \left| \frac{y_i}{|y|^{\alpha+2}} \right|^p \mathrm{d}y \\
&\leq 2^{-kp} \alpha^p \int_{B_2(0)} |y|^{-(\alpha+1)p} \,\mathrm{d}y && (|y_i| \leq |y|) \\
&\leq 2^{-kp} \alpha^p A_N \int_0^2 r^{-(\alpha+1)p}\ r^{N-1} \,\mathrm{d}r \\
&\leq 2^{-kp}\ A_N \frac{\alpha^p 2^{N-\alpha p - p}}{N - \alpha p - p} && (N - \alpha p - p > 0)\,. \qquad (2)
\end{aligned}$$

From Eqs. (1) and (2), we obtain

$$\|f_k\|_{\mathrm{W}^{1,p}(B_1(0))} \leq \frac{C}{2^k} \quad \text{with} \quad C = \left[A_N \frac{2^{N-\alpha p}}{N - \alpha p} + N A_N \frac{\alpha^p 2^{N-\alpha p - p}}{N - \alpha p - p} \right]^{\frac{1}{p}}$$

so the functions f_k are in $\mathrm{W}^{1,p}(B_1(0))$, for all $k \in \mathbb{N}$. Therefore, using the triangular inequality in $\mathrm{W}^{1,p}(B_1(0))$, we obtain

$$\|f\|_{\mathrm{W}^{1,p}(B_1(0))} \leq \sum_{k=0}^{+\infty} \|f_k\|_{\mathrm{W}^{1,p}(B_1(0))} \leq \sum_{k=0}^{+\infty} \frac{C}{2^k} = 2C$$

so the series converge absolutely in $\mathrm{W}^{1,p}(B_1(0))$ and the function f is in $\mathrm{W}^{1,p}(B_1(0))$.

However, the function f is unbounded on all open subset $U \subset \Omega$ meaning that $f \notin \mathrm{L}^\infty(U)$. By definition of the denseness of the family $(r_k)_{k\in\mathbb{N}}$ there exist $k_0 \in \mathbb{N}$ such that $r_{k_0} \in U$, so for all $T > 0$, we have

$$f^{-1}\left((T,+\infty)\right) \supset B_t\left(r_{k_0}\right) \cap \omega \quad \text{with } t = \left(2^{k_0} T\right)^{-\frac{1}{\alpha}}$$

and the open set $B_t\left(r_{k_0}\right) \cap U$ has positive Lebesgue measure therefore $f \notin \mathrm{L}^\infty(U)$[1].

Exercise 3.

For a Lipschitz domain $\Omega \subset \mathbb{R}^N$ and a function $A \in C^1(\Omega; \mathbb{R}^{N\times N})$, find the weak formulation for

$$\begin{cases} -\operatorname{div}(A\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}. \tag{3}$$

Solution 3.

Take $\phi \in C_0^\infty(\Omega)$ then multiply and integrate Eq. (3) which give

$$-\int_\Omega \operatorname{div}(A\nabla u)(x)\,\phi(x)\,\mathrm{d}x = \int_\Omega f(x)\,\phi(x)\,\mathrm{d}x,$$

using the divergence theorem on the left hand side, we obtain

$$-\int_\Omega \operatorname{div}(A\nabla u)(x)\,\phi(x)\,\mathrm{d}x = \int_\Omega (A\nabla u(x))\cdot\nabla\phi(x)\,\mathrm{d}x - \int_{\partial\Omega} \partial_\nu u(x)\,\phi(x)\,\mathrm{d}\sigma(x).$$

Finally, we obtain

$$\int_\Omega (A\nabla u(x))\cdot\nabla\phi(x)\,\mathrm{d}x = \int_\Omega f(x)\,\phi(x)\,\mathrm{d}x. \tag{4}$$

Remark 1. As we see from Eq. (4) that the weak form has the advantage that it make sense for functions with low regularity (not C^k), for example $u, \phi \in \mathrm{H}^1(\Omega)$, $A \in \mathrm{L}^\infty(\Omega; \mathbb{R}^{N\times N})$, and $f \in \mathrm{L}^2(\Omega)$. I would like to emphasise that for Eq. (4) to make sense, we do **not** need to define the second derivative of the functions u and ϕ.

When we speak about the operator $u \mapsto \operatorname{div}(A\nabla u)$ with a rough $A \in \mathrm{L}^\infty(\Omega; \mathbb{R}^{N\times N})$, the expresion $\operatorname{div}(A\nabla u)$ is to be understood in the sense of the weak form.

Exercise 4.

Let X, Y be Banach spaces and assume that $X_0 \subset X$ is a dense subset. Furthermore assume that there is a linear operator $T_0 : X_0 \to Y$ such that $\|T_0 f\|_Y \leq C\,\|f\|_X$ for all $f \in X_0$. Show that there is a unique bounded linear operator $T : X \to Y$ such that $T|_{X_0} = T_0$ with operator norm $\leq C$.

Solution 4.

Let's first establish that for a sequence $(f_n)_{n\in\mathbb{N}} \in X_0^{\mathbb{N}}$ such that $f_n \to f \in X$, the sequence $(T_0 f_n)_{n\in\mathbb{N}}$ is convergent in Y. Indeed, the sequence $(f_n)_{n\in\mathbb{N}}$ converge in X therefore is a Cauchy sequence and with $\|T_0(f_n - f_m)\|_Y \leq C\,\|f_n - f_m\|_X$, the sequence $(T_0 f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y which imply that the sequence $(T_0 f_n)_{n\in\mathbb{N}}$ converge in Y.

We define $T : X \to Y$ by

$$Tf = \lim_{n\to+\infty} T_0 f_n \quad \text{where } (f_n)_{n\in\mathbb{N}} \in X_0^{\mathbb{N}} \text{ such that } f_n \to f.$$

[1] Recall that $\|f\|_\infty \leq T$ if, and only if, $f^{-1}((T,+\infty))$ has zero Lebesgue measure.

By definition, we have $T|_{X_0} = T_0$. For the operator norm, for $f \in X$ and $(f_n)_{n\in\mathbb{N}} \in X_0^{\mathbb{N}}$ such that $f_n \to f$, we compute

$$\|Tf\|_Y = \lim_{n\to+\infty} \|T_0 f_n\|_Y \leq C \lim_{n\to+\infty} \|f_n\|_X = C\,\|f\|_X$$

which give that the operator norm of T is $\leq C$.

For the uniqueness, let's assume that there exists $T, S : X \to Y$ such that $T|_{X_0} = S|_{X_0} = T_0$, for $f \in X$ and $(f_n)_{n\in\mathbb{N}} \in X_0^{\mathbb{N}}$ such that $f_n \to f$, we compute

$$(T - S)\,f = \lim_{n\to+\infty} (T - S)\,f_n = \lim_{n\to+\infty} T_0 f_n - T_0 f_n = 0$$

which imply that $T = S$.

Example 1. Show that for $I = (a, b) \subset \mathbb{R}$ there exists a constant $C > 0$ such that for all $u \in \mathrm{W}_0^{1,p}(I)$, we have

$$\|u\|_p \leq C\,\|u'\|_p\,. \qquad \text{(Poincaré's inequality)}$$

For $u \in C_0^\infty(I)$, we write $u(x) = \int_a^x u'(x)\,\mathrm{d}x$ and we compute

$$|u(x)| \leq \int_a^x |u'(x)|\,\mathrm{d}x \leq \int_a^b |u'(x)|\,\mathrm{d}x \leq |b-a|^{1-\frac{1}{p}}\,\|u'\|_p$$

the last inequality comes from the Hölder's inequality. Then, we obtain

$$\|u\|_p \leq |b-a|\,\|u'\|_p \qquad \text{for } u \in C_0^\infty(I).$$

Finally, by density the inequality remain true on $\mathrm{W}_0^{1,p}(I)$ because each side is continues in $\mathrm{W}_0^{1,p}(I)$.