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Sobolev spaces: tutorials

Exercise sheet 02

Exercise 1.
For an open set $\Omega \subset \mathbb{R}^N$, constants $a, b > 0$, and a function $f \in \mathrm{L}^2(\Omega)$, consider the problem: find $u \in \mathrm{L}^2(\Omega)$ such that

$$\begin{cases} \Delta^2 u - a\Delta u + bu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where ∂_ν is the normal exterior derivative.

1. Prove for $u, v \in \mathrm{H}_0^2(\Omega)$ the identity
$$\int_\Omega \Delta u(x)\, \Delta v(x)\, \mathrm{d}x = \sum_{i,j=1}^{N} \int_\Omega \partial_{i,j} u(x)\, \partial_{i,j} v(x)\, \mathrm{d}x.$$
2. Show that the weak formulation of Eq. (1) is
$$\int_\Omega \Delta u(x)\Delta v(x) + a\,\nabla u(x) \cdot \nabla v(x) + b\, u(x)v(x)\, \mathrm{d}x = \int_\Omega f(x)v(x)\, \mathrm{d}x, \quad \forall u, v \in \mathrm{H}_0^2(\Omega).$$
3. Show that Eq. (1) has a unique weak solution $u \in \mathrm{H}_0^2(\Omega)$ using the Lax-Milgram Lemma.

Exercise 2.
For $f, g, h \in C_0^\infty(\mathbb{R}^N)$ show the following statements:

1. The convolution as binary operation is commutative, meaning $f * g = g * f$.
2. The convolution as binary operation is associative, meaning $(f * g) * h = f * (g * h)$.
3. For a function g, we note $\breve{g}(x) = g(-x)$ for all $x \in \mathbb{R}^N$, we have
$$\int_{\mathbb{R}^N} (f * g)(x)\, h(x)\, \mathrm{d}x = \int_{\mathbb{R}^N} f(x)\, (\breve{g} * h)(x)\, \mathrm{d}x. \tag{2}$$

Now assume $f \in \mathrm{W}^{k,p}(\mathbb{R}^N)$ and $g \in \mathrm{L}^q(\mathbb{R}^N)$ for some $k \in \mathbb{N}$, $1 \le p \le \infty$, and $1 \le q < \infty$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

4. Show that $f * g \in \mathrm{W}^{k,r}(\mathbb{R}^N)$ by proving $\partial^\alpha (f * g) = (\partial^\alpha f) * g$ for all multi-indices $\alpha \in \mathbb{N}^N$ such that $|\alpha| \le k$. (hint: density)
5. Show that $\operatorname{supp}(f * g) \subset \overline{\operatorname{supp}(f) + \operatorname{supp}(g)}$ where
$$\operatorname{supp}(f) = \mathbb{R}^N \setminus \bigcup \left\{ \Omega \subset \mathbb{R}^N \;\middle|\; \Omega \text{ open and } f = 0 \text{ almost everywhere in } \Omega \right\}.$$