Karlsruhe Institute of Technology

Institute for Analysis
PD Dr. Rainer Mandel
Dr. Zoïs Moitier

May 13, 2021

Sobolev spaces: tutorials

Exercise sheet 02 with solution

Exercise 1.
For an open set $\Omega \subset \mathbb{R}^N$, constants $a, b > 0$, and a function $f \in \mathrm{L}^2(\Omega)$, consider the problem: find $u \in \mathrm{L}^2(\Omega)$ such that

$$\begin{cases} \Delta^2 u - a\Delta u + bu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where ∂_ν is the normal exterior derivative.

1. Prove for $u, v \in \mathrm{H}_0^2(\Omega)$ the identity

$$\int_\Omega \Delta u(x)\, \Delta v(x)\, \mathrm{d}x = \sum_{i,j=1}^N \int_\Omega \partial_{i,j} u(x)\, \partial_{i,j} v(x)\, \mathrm{d}x.$$

2. Show that the weak formulation of Eq. (1) is

$$\int_\Omega \Delta u(x)\Delta v(x) + a\, \nabla u(x) \cdot \nabla v(x) + b\, u(x)v(x)\, \mathrm{d}x = \int_\Omega f(x)v(x)\, \mathrm{d}x, \quad \forall u, v \in \mathrm{H}_0^2(\Omega).$$

3. Show that Eq. (1) has a unique weak solution $u \in \mathrm{H}_0^2(\Omega)$ using the Lax-Milgram Lemma.

Solution 1.
Question 1. Take $u, v \in \mathrm{H}_0^2(\Omega)$, using $\mathrm{H}_0^2(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{2,2}}$, there exists a sequence $(v_n)_{n\in\mathbb{N}} \in C_0^\infty(\Omega)^{\mathbb{N}}$ such that $\|v_n - v\|_{2,2} \to 0$ as $n \to +\infty$. We compute

$$\begin{aligned} \int_\Omega \Delta u(x)\, \Delta v(x)\, \mathrm{d}x &= \int_\Omega \left(\sum_{i=1}^N \partial_{i,i} u(x)\right) \left(\sum_{j=1}^N \partial_{j,j} v(x)\right) \mathrm{d}x \\ &= \sum_{i,j=1}^N \int_\Omega \partial_{i,i} u(x)\, \partial_{j,j} v(x)\, \mathrm{d}x \\ &\overset{(1)}{=} \sum_{i,j=1}^N \lim_{n\to+\infty} \int_\Omega \partial_{i,i} u(x)\, \partial_{j,j} v_n(x)\, \mathrm{d}x \\ &= -\sum_{i,j=1}^N \lim_{n\to+\infty} \int_\Omega \partial_i u(x)\, \partial_{i,j,j} v_n(x)\, \mathrm{d}x \qquad \left(\int_{\partial\Omega} \partial_i u\, \partial_{j,j} v_n\, \nu_i\, \mathrm{d}\sigma_x = 0\right) \end{aligned}$$

$$= \sum_{i,j=1}^{N} \lim_{n\to+\infty} \int_\Omega \partial_{i,j} u(x)\, \partial_{i,j} v_n(x)\, \mathrm{d}x \qquad \left(\int_{\partial\Omega} \partial_i u\, \partial_{i,j} v_n\, \nu_j\, \mathrm{d}\sigma_x = 0\right)$$

$$\overset{(2)}{=} \sum_{i,j=1}^{N} \int_\Omega \partial_{i,j} u(x)\, \partial_{i,j} v(x)\, \mathrm{d}x.$$

The equalities (1) and (2) are justified if, for $w \in \mathrm{L}^2(\Omega)$ and $1 \le i,j \le N$, the following limit hold

$$\int_\Omega w(x)\, \partial_{i,j} v(x)\, \mathrm{d}x = \lim_{n\to+\infty} \int_\Omega w(x)\, \partial_{i,j} v_n(x)\, \mathrm{d}x. \tag{2}$$

We compute

$$\begin{aligned} \left| \int_\Omega w(x)\, \partial_{i,j} v(x)\, \mathrm{d}x - \int_\Omega w(x)\, \partial_{i,j} v_n(x)\, \mathrm{d}x \right| &\le \int_\Omega |w(x)|\, |\partial_{i,j} v_n(x) - \partial_{i,j} v(x)|\, \mathrm{d}x \\ &\le \|w\|_2\, \|\partial_{i,j} v - \partial_{i,j} v_n\|_2 \end{aligned}$$

and since $\|\partial_{i,j} v - \partial_{i,j} v_n\|_2 \to 0$, as $n \to +\infty$, by the definition of the sequence $(v_n)_{n\in\mathbb{N}}$, we get Eq. (2).

Question 2. We multiply Eq. (1) by $v \in \mathrm{H}_0^2(\Omega)$ and integrate on Ω to get

$$\int_\Omega \Delta^2 u(x)\, v(x) - a\, \Delta u(x)\, v(x) + b\, u(x)\, v(x)\, \mathrm{d}x = \int_\Omega f(x)\, v(x)\, \mathrm{d}x$$

then using the divergence theorem, we get

$$\begin{aligned} \int_\Omega \Delta^2 u(x)\, v(x)\, \mathrm{d}x &= -\int_\Omega \nabla \Delta u(x) \cdot \nabla v(x)\, \mathrm{d}x && \left(\int_{\partial\Omega} \partial_\nu \Delta u\, v\, \mathrm{d}\sigma_x = 0\right) \\ &= \int_\Omega \Delta u(x)\, \Delta v(x)\, \mathrm{d}x && \left(\int_{\partial\Omega} \Delta u\, \partial_\nu v\, \mathrm{d}\sigma_x = 0\right) \\ \int_\Omega \Delta u(x)\, v(x)\, \mathrm{d}x &= \int_\Omega \nabla u(x) \cdot \nabla v(x)\, \mathrm{d}x && \left(\int_{\partial\Omega} \partial_\nu u\, v\, \mathrm{d}\sigma_x = 0\right) \end{aligned}$$

which give

$$\int_\Omega \Delta u(x) \Delta v(x) + a\, \nabla u(x) \cdot \nabla v(x) + b\, u(x) v(x)\, \mathrm{d}x = \int_\Omega f(x) v(x)\, \mathrm{d}x, \quad \forall v \in \mathrm{H}_0^2(\Omega).$$

Question 3. We verify the hypothesis of the Lax-Milgram theorem on the Hilbert space $\mathrm{H}_0^2(\Omega)$ with the bilinear form $d : \mathrm{H}_0^2(\Omega) \times \mathrm{H}_0^2(\Omega) \to \mathbb{R}$ and linear form $\ell : \mathrm{H}_0^2(\Omega) \to \mathbb{R}$ given by

$$\begin{aligned} d(u,v) &= \int_\Omega \Delta u(x) \Delta v(x) + a\, \nabla u(x) \cdot \nabla v(x) + b\, u(x) v(x)\, \mathrm{d}x, \\ \ell(v) &= \int_\Omega f(x) v(x)\, \mathrm{d}x. \end{aligned}$$

(i) d is bounded: for $u, v \in \mathrm{H}_0^2(\Omega)$, we have

$$|d(u,v)| \le \|\Delta u\|_2\, \|\Delta v\|_2 + a\, \|\nabla u\|_2\, \|\nabla v\|_2 + b\, \|u\|_2\, \|v\|_2 \le (N + Na + b)\, \|u\|_{2,2}\, \|v\|_{2,2}$$

because $\|\Delta u\|_2 \le N\, \|u\|_{2,2}$, $\|\nabla u\|_2 \le N\, \|u\|_{2,2}$, and $\|u\|_2 \le \|u\|_{2,2}$.

(ii) d is coercive: for $u \in \mathrm{H}_0^2(\Omega)$, we have

$$d(u,u) = \|\Delta u\|_2^2 + a\,\|\nabla u\|_2^2 + b\,\|u\|_2^2\,.$$

Using *Question 1*, we get

$$\|\Delta u\|_2^2 = \int_\Omega |\Delta u(x)|^2\,\mathrm{d}x = \sum_{i,j=1}^N \int_\Omega |\partial_{i,j} u(x)|^2\,\mathrm{d}x = \sum_{\substack{\alpha\in\mathbb{N}^N\\|\alpha|=2}} \|\partial^\alpha u\|_2^2$$

and rewriting the gradient term

$$\|\nabla u\|_2^2 = \int_\Omega |\nabla u(x)|^2\,\mathrm{d}x = \int_\Omega \sum_{i=1}^N |\partial_i u(x)|^2\,\mathrm{d}x = \sum_{\substack{\alpha\in\mathbb{N}^N\\|\alpha|=1}} \|\partial^\alpha u\|_2^2\,.$$

Therefore, we get

$$d(u,u) \ge \min(1,a,b)\,\|u\|_2^2\,.$$

(iii) ℓ is bounded: for $v \in \mathrm{H}_0^2(\Omega)$, we have

$$|\ell(v)| \le \|f\|_2\,\|v\|_2 \le \|f\|_2\,\|v\|_{2,2}\,.$$

Using the Lax-Milgram theorem there exists an unique $u \in \mathrm{H}_0^2(\Omega)$ such that $d(u,v) = \ell(v)$ for all $v \in \mathrm{H}_0^2(\Omega)$ and $\|u\|_{2,2} \le \max\left(1, a^{-1}, b^{-1}\right)\|f\|_2$.

Exercise 2.
For $f, g, h \in C_0^\infty(\mathbb{R}^N)$ show the following statements:

1. The convolution as binary operation is commutative, meaning $f * g = g * f$.
2. The convolution as binary operation is associative, meaning $(f * g) * h = f * (g * h)$.
3. For a function g, we note $\breve{g}(x) = g(-x)$ for all $x \in \mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} (f * g)(x)\,h(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} f(x)\,(\breve{g} * h)(x)\,\mathrm{d}x. \tag{3}$$

Now assume $f \in \mathrm{W}^{k,p}(\mathbb{R}^N)$ and $g \in \mathrm{L}^q(\mathbb{R}^N)$ for some $k \in \mathbb{N}$, $1 \le p \le \infty$, and $1 \le q < \infty$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

4. Show that $f * g \in \mathrm{W}^{k,r}(\mathbb{R}^N)$ by proving $\partial^\alpha(f * g) = (\partial^\alpha f) * g$ for all multi-indices $\alpha \in \mathbb{N}^N$ such that $|\alpha| \le k$. (hint: density)
5. Show that $\operatorname{supp}(f * g) \subset \overline{\operatorname{supp}(f) + \operatorname{supp}(g)}$ where

$$\operatorname{supp}(f) = \mathbb{R}^N \setminus \bigcup \left\{\Omega \subset \mathbb{R}^N \;\middle|\; \Omega \text{ open and } f = 0 \text{ almost everywhere in } \Omega\right\}.$$

Solution 2.

Question 1. Using the change of variable $z = x - y$, we get

$$(f * g)(x) = \int_{\mathbb{R}^N} f(x-y)\, g(y)\, \mathrm{d}y = \int_{\mathbb{R}^N} f(z)\, g(x-z)\, \mathrm{d}z = (g * f)(x).$$

Remark 1. The commutative property of the convolution stay true for $f \in \mathrm{L}^p(\mathbb{R}^N)$ and $g \in \mathrm{L}^q(\mathbb{R}^N)$ with $1 \le p, q \le \infty$ such that $\frac{1}{p} + \frac{1}{q} \ge 1$.

Question 2. We compute

$$\begin{aligned}
((f * g) * h)(x) &= \int_{\mathbb{R}^N} (f * g)(x-y)\, h(y)\, \mathrm{d}y \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x-y-z)\, g(z)\, \mathrm{d}z\, h(y)\, \mathrm{d}y \\
&= \int_{\mathbb{R}^N \times \mathbb{R}^N} f(x-y-z)\, g(z)\, h(y)\, \mathrm{d}z\, \mathrm{d}y \qquad (z = w - y \text{ and } \mathrm{d}z = \mathrm{d}w) \\
&= \int_{\mathbb{R}^N \times \mathbb{R}^N} f(x-w)\, g(w-y)\, h(y)\, \mathrm{d}w\, \mathrm{d}y \\
&= \int_{\mathbb{R}^N} f(x-w) \int_{\mathbb{R}^N} g(w-y)\, h(y)\, \mathrm{d}y\, \mathrm{d}w \\
&= \int_{\mathbb{R}^N} f(x-w)\, (g * h)(w)\, \mathrm{d}w \\
&= (f * (g * h))(x)
\end{aligned}$$

all the swap of integrals are valid by Fubini's theorem.

Remark 2. The associative properties of the convolution stay true for $f \in \mathrm{L}^p(\mathbb{R}^N)$, $g \in \mathrm{L}^q(\mathbb{R}^N)$, and $h \in \mathrm{L}^r(\mathbb{R}^N)$ with $1 \le p, q, r \le \infty$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \ge 2$.

Question 3. We compute

$$\begin{aligned}
\int_{\mathbb{R}^N} (f * g)(x)\, h(x)\, \mathrm{d}x &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x-y) g(y)\, \mathrm{d}y\, h(x)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^N \times \mathbb{R}^N} f(x-y) g(y) h(x)\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_{\mathbb{R}^N \times \mathbb{R}^N} f(z) g(x-z) h(x)\, \mathrm{d}x\, \mathrm{d}z \qquad (z = x - y \text{ and } \mathrm{d}z = -\,\mathrm{d}y) \\
&= \int_{\mathbb{R}^N \times \mathbb{R}^N} f(z) \breve{g}(z-x) h(x)\, \mathrm{d}x\, \mathrm{d}z \qquad (\breve{g}(x) = g(-x)) \\
&= \int_{\mathbb{R}^N} f(z) \int_{\mathbb{R}^N} \breve{g}(z-x) h(x)\, \mathrm{d}x\, \mathrm{d}z \\
&= \int_{\mathbb{R}^N} f(z)\, (\breve{g} * h)(z)\, \mathrm{d}z.
\end{aligned}$$

Remark 3. The Eq. (3) stay true for $f \in \mathrm{L}^p(\mathbb{R}^N)$, $g \in \mathrm{L}^q(\mathbb{R}^N)$, and $h \in \mathrm{L}^r(\mathbb{R}^N)$ with $1 \le p, q, r \le \infty$ such that $1 \le \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \le 2$.

Question 4. **Step 1**: First, we start by showing that for $f \in \mathrm{L}^p(\mathbb{R}^N)$, $1 \le p \le \infty$, and $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have $f * \varphi \in C^\infty(\mathbb{R}^N)$ which is needed to show the Proposition 4.7 in the lecture note.

We start by showing that for $f \in \mathrm{L}^p(\mathbb{R}^N)$ and $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have $\partial_i (f * \varphi) = f * (\partial_i \varphi)$ for $i \in \{1, \dots, N\}$. We fix $x \in \mathbb{R}^N$, $\rho > 0$, and $i \in \{1, \dots, N\}$. We observe that

$$f * \varphi(x) = \int_{\mathbb{R}^N} \varphi(x-y) f(y) \, \mathrm{d}y = \int_K \varphi(x-y) f(y) \, \mathrm{d}y$$

where K is a compact set such that $x - \mathrm{supp}(\varphi) \subset K$, for all $x \in B(x, \rho)$. Let e_i be the i-nth base vector, for ε small enough such that $x + \varepsilon e_i \in B(x, \rho)$, we compute

$$\frac{f * \varphi(x + \varepsilon e_i) - f * \varphi(x)}{\varepsilon} = \int_K \frac{\varphi(x - y + \varepsilon e_i) - \varphi(x-y)}{\varepsilon} f(y) \, \mathrm{d}y.$$

The quantity

$$\frac{\varphi(x - y + \varepsilon e_i) - \varphi(x-y)}{\varepsilon} f(y) \longrightarrow \partial_i \varphi(x-y) \, f(y) \quad \text{as } \varepsilon \to 0$$

for almost every $y \in K$, the convergence is uniform, and, for almost every $y \in \mathbb{R}^N$ and ε small enough, we have

$$\left| \frac{\varphi(x - y + \varepsilon e_i) - \varphi(x-y)}{\varepsilon} f(y) \right| \leq \sup_{z \in \mathrm{supp}(\varphi)} |\partial_i \varphi(z)| \, |f(y)| \, \mathbb{1}_K(y) =: V(y)$$

with $V \in \mathrm{L}^1(\mathbb{R}^N)$ because $\|V\|_1 \leq \lambda(K)^{1 - \frac{1}{p}} \|f\|_p$, by the Hölder inequality, for $\lambda(K)$ the measure of K. Using the dominated convergence theorem, we obtain

$$\partial_i \int_\Omega \varphi(x-y) \, f(y) \, \mathrm{d}y = \int_\Omega \partial_i \varphi(x-y) \, f(y) \, \mathrm{d}x.$$

Since this is true for any ball in $\mathbb{R}^N$, it is true everywhere, $\partial_i (f * \varphi)(x) = f * (\partial_i \varphi)(x)$ for $x \in \mathbb{R}^N$.

Then, by induction on $|\alpha|$, for $f \in \mathrm{L}^p(\mathbb{R}^N)$ and $\varphi \in C_0^\infty(\mathbb{R}^N)$ we get $\partial^\alpha (f * \varphi) = f * (\partial^\alpha \varphi)$ for all multi-indices $\alpha \in \mathbb{N}^N$. We want to emphasize that only this part use to show the density result Theorem 4.8 in the lecture note.

Step 2: We show that, for $f \in \mathrm{W}^{k,p}(\mathbb{R}^N)$, $k \in \mathbb{N}$, $1 \leq p \leq \infty$, and $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have $\partial^\alpha (f * \varphi) = (\partial^\alpha f) * \varphi$ for $\alpha \in \mathbb{N}^N$ multi-index such that $|\alpha| \leq k$. We compute

$$\begin{aligned}
\partial^\alpha (f * \varphi) &= f * (\partial^\alpha \varphi) && (\textbf{Step 1}) \\
&= \int_{\mathbb{R}^N} f(x-y) \, \partial_y^\alpha \varphi(y) \, \mathrm{d}y \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^N} \partial_y^\alpha f(x-y) \, \varphi(y) \, \mathrm{d}y && (\text{def. weak derivative}) \\
&= (-1)^{2|\alpha|} \int_{\mathbb{R}^N} \partial_x^\alpha f(x-y) \, \varphi(y) \, \mathrm{d}y \\
&= (\partial^\alpha f) * \varphi
\end{aligned}$$

where ∂_x^α (resp. ∂_y^α) is the derivative with respect to x (resp. y).

Step 3: Now, for $1 \leq p \leq \infty$, $1 \leq q < \infty$, and $1 \leq r \leq \infty$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $f \in \mathrm{W}^{k,p}(\mathbb{R}^N)$, and $g \in \mathrm{L}^q(\mathbb{R}^N)$, we want to show that $\partial^\alpha (f * g) = (\partial^\alpha f) * g$ for all multi-indices $\alpha \in \mathbb{N}^N$ such that $|\alpha| \leq k$. Using Theorem 4.8 in the lecture, there exists $(\varphi_n)_{n \in \mathbb{N}} \in C_0^\infty(\mathbb{R}^N)^{\mathbb{N}}$

such that $\varphi_n \to g$ as $n \to +\infty$ in $\mathrm{L}^q(\mathbb{R}^N)$. For $\psi \in C_0^\infty(\mathbb{R}^N)$, we compute the ∂^α weak derivative of $f * g$,

$$\begin{aligned}
\int_{\mathbb{R}^N} f * g(x)\, \partial^\alpha \psi(x)\, \mathrm{d}x &\overset{(1)}{=} \lim_{n \to +\infty} \int_{\mathbb{R}^N} f * \varphi_n(x)\, \partial^\alpha \psi(x)\, \mathrm{d}x \\
&= \lim_{n \to +\infty} (-1)^{|\alpha|} \int_{\mathbb{R}^N} \partial^\alpha (f * \varphi_n)(x)\, \psi(x)\, \mathrm{d}x \quad \text{(def. weak derivative)} \\
&= \lim_{n \to +\infty} (-1)^{|\alpha|} \int_{\mathbb{R}^N} (\partial^\alpha f) * \varphi_n(x)\, \psi(x)\, \mathrm{d}x \quad (\textbf{Step 2}) \\
&\overset{(2)}{=} (-1)^{|\alpha|} \int_{\mathbb{R}^N} (\partial^\alpha f) * g(x)\, \psi(x)\, \mathrm{d}x
\end{aligned}$$

therefore the ∂^α weak derivative of $f * g$ is $(\partial^\alpha f) * g$. The equality (1) is justify by

$$\left| \int_{\mathbb{R}^N} f * (g - \varphi_n)(x)\, \partial^\alpha \psi(x)\, \mathrm{d}x \right| \leq \|f\|_p \|g - \varphi_n\|_q \|\partial^\alpha \psi\|_s$$

and the equality (2) is justify by

$$\left| \int_{\mathbb{R}^N} (\partial^\alpha f) * (g - \varphi_n)(x)\, \psi(x)\, \mathrm{d}x \right| \leq \|\partial^\alpha f\|_p \|g - \varphi_n\|_q \|\psi\|_s$$

with $\frac{1}{r} + \frac{1}{s} = 1$. Using this weak derivative, we get that

$$\|\partial^\alpha (f * g)\|_r = \|(\partial^\alpha f) * g\|_r \leq \|\partial^\alpha f\|_p \|q\|_q$$

therefore we obtain $f * g \in \mathrm{W}^{k,p}(\mathbb{R}^N)$.

Remark 4. Using this question and the commutative property of the convolution, we can show that $\mathrm{W}^{k,p}(\mathbb{R}^N) * \mathrm{W}^{\ell,q}(\mathbb{R}^N) \subset \mathrm{W}^{k+\ell,r}(\mathbb{R}^N)$, for $k, \ell \geq 0$ and $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Therefore, the convolution add up the regularities of the two functions.

Question 5. We show that

$$\overline{\operatorname{supp}(f) + \operatorname{supp}(g)}^{\complement} \subset \operatorname{supp}(f * g)^{\complement} = \bigcup \left\{ \Omega \subset \mathbb{R}^N \;\middle|\; \Omega \text{ open and } f * g = 0 \text{ a.e. in } \Omega \right\}. \tag{4}$$

For $x \in \overline{\operatorname{supp}(f) + \operatorname{supp}(g)}^{\complement}$, there exists $r > 0$ such that $B(x, r) \subset \overline{\operatorname{supp}(f) + \operatorname{supp}(g)}^{\complement}$. For $y \in B(x, r)$, we have

$$(f * g)(y) = \int_{\mathbb{R}^N} f(y - z)\, g(z)\, \mathrm{d}z = \int_{\operatorname{supp}(g)} f(y - z)\, g(z)\, \mathrm{d}z.$$

However, for $y \in B(x, r)$ and $z \in \operatorname{supp}(g)$, if we had

$$y - z \in \operatorname{supp}(f) \implies y \in \operatorname{supp}(f) + \{z\} \subset \operatorname{supp}(f) + \operatorname{supp}(g)$$

which contradict $B(x, r) \subset \overline{\operatorname{supp}(f) + \operatorname{supp}(g)}^{\complement}$ therefore $y - z \notin \operatorname{supp}(f)$ and we obtain that $(f * g)(y) = 0$ for all $y \in B(x, r)$. So, we have the inclusion Eq. (4) and by taking the complement we get the question statement.