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Sobolev spaces: tutorials

Exercise sheet 03

Exercise 1.
Take a bounded connected open set $\Omega \subset \mathbb{R}^N$ and a function $u \in \mathrm{W}^{k,p}(\Omega)$ with $k \geq 1$ and $1 \leq p \leq \infty$, show that:

1. If $\nabla u = 0$ then u is constant.
2. If, for all multi-indices $|\alpha| = k$, we have $\partial^\alpha u = 0$ then u is a polynomial of total degree[1] less or equal to $k-1$.

Exercise 2.
We define the open half-space $\mathbb{R}^N_+ := \left\{(x_1, \dots, x_N) \in \mathbb{R}^N \;\middle|\; x_N > 0\right\}$ and an extension operator $E : \mathrm{L}^p(\mathbb{R}^N_+) \to \mathrm{L}^p(\mathbb{R}^N)$ define by

$$Eu(x) = \begin{cases} u(x) & \text{if } x \in \mathbb{R}^N_+ \\ u(x', -x_N) & \text{if } x' \in \mathbb{R}^{N-1} \text{ and } x_N < 0 \end{cases}.$$

1. Show that $E : \mathrm{W}^{1,p}(\mathbb{R}^N_+) \to \mathrm{W}^{1,p}(\mathbb{R}^N)$ is a linear continuous operator.
2. Show that there exists $u \in \mathrm{W}^{2,p}(\mathbb{R}^N_+)$ such that $Eu \notin \mathrm{W}^{2,p}(\mathbb{R}^N)$. Search u of the form $u(x) = \varphi(x')\, f(x_N)$ with $\varphi \in C_0^\infty(\mathbb{R}^{N-1})$ and $f \in C^2(\mathbb{R}_+)$ with $f'(0) \neq 0$.

Now, we define a extension operator $F : \mathrm{L}^p(\mathbb{R}^N_+) \to \mathrm{L}^p(\mathbb{R}^N)$ define by

$$Fu(x) = \begin{cases} u(x) & \text{if } x_N > 0 \\ a\, u(x', -\lambda\, x_N) + b\, u(x', -\mu\, x_N) & \text{if } x_N < 0 \end{cases}$$

with $a, b \in \mathbb{R}$ and $\lambda, \mu \in \mathbb{R}_+$.

3. For a multi-index $\alpha \in \mathbb{N}^N$ such that $|\alpha| \leq 2$, we define v_α by

$$v_\alpha(x) = \begin{cases} \partial^\alpha u(x) & \text{if } x_N > 0 \\ a(-\lambda)^{\alpha_N}\, \partial^\alpha u(x', -\lambda\, x_N) + b(-\mu)^{\alpha_N}\, \partial^\alpha u(x', -\mu\, x_N) & \text{if } x_N < 0 \end{cases}.$$

   Show that $\partial^\alpha(Fu) = v_\alpha$ if, and only if,

$$\begin{cases} a + b = 1 \\ -\lambda a - \mu b = 1 \end{cases}.$$

4. Show that there exist $a, b \in \mathbb{R}$ and $\lambda, \mu \in \mathbb{R}_+$ such that $F : \mathrm{W}^{2,p}(\mathbb{R}^N_+) \to \mathrm{W}^{2,p}(\mathbb{R}^N)$ is a linear continuous operator.

[1] We recall that a multivariate polynomial of total degree k is given by $P(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha$ where $c_\alpha \in \mathbb{R}$ and $x^\alpha = x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ for all multi-indices $\alpha \in \{0, \dots, k\}^N$.